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EDITED BY

W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF

F. S. MACAULAY, M.A., D.Sc.

AND

Prof. E. T. WHITTAKER, Sc.D., F.R.S.

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VOL. XIII. MAY, 1926. No. 182.

ASYMPTOTES.

BY PROF. T. P. NUNN, M.A., D.Sc.

ACCORDING to the standard-text-books, an asymptote is a line which meets a curve at two points at infinity, but is not wholly at infinity. Regarded as a flash of mathematical wit the definition is clever and illuminating, but it is far too subtle for a raw student. He may have learnt that to call parallel lines "lines that meet in a point at infinity" is merely a Pickwickian way of saying that they do not meet at all, but, being in a plane, had (so to speak) their chance of meeting and just failed to take it; but even if he understands that, he may still be mystified here. For the asymptote and the curve meet, he is told, at *two* points at infinity, and who could guess what complicated tragedy of failure that euphemism conceals?

One may write flippantly, but the matter is really serious. Language like the definition of an asymptote occurs often in the text-books, and always tends to puzzle and confuse the honest student who believes that mathematical statements, above all others, mean exactly what they say. In the case of a subject which we profess to teach largely because it fosters clear thinking and creates an appetite for it, this result is peculiarly unfortunate. But before inquiring how it may be avoided, we must point to its source.

For this purpose consider the equations $ax+by+c=0$ and $F=0$ and the corresponding loci, F being a function of the second degree involving the first and second powers of the variables together with constants. It is usual to say that we can, by means of certain simple algebraic processes, find two pairs of values of x and y which satisfy both these equations at once; but that statement is, of course, inexact. What the processes really yield are two pairs of *formulae* from which, when numerical values are assigned to the constants, numerical values of x and y satisfying the equations simultaneously may be deduced if such values exist. For the formulae are of such a character that they do not yield numerical values (in the ordinary sense of "numerical") unless certain conditions obtain among the constants. For instance, the attempt to determine x from one of the formulae may, with a particular set of constants, lead to the stage $x=-4/0$, and with another set to the stage $x=+3\pm\sqrt{-5}$; and it is not possible to proceed from either of these stages to the calculation of a single number which is the value of x. The natural conclusion would be that the given equations sometimes are and sometimes

are not capable of being satisfied by the same pair of numerical values of x and y. But to admit this would be contrary to the spirit of mathematics, which always aims at generalisations unqualified by exceptions. Since there are two *formal* solutions of the problem of finding a pair of values of the variables which satisfy both of the equations, the mathematician postulates that, for every set of values of the constants, there are also two *actual* solutions; and in order to make the postulate good, extends the notion of number to include such forms as $-4/0$ and $+3\pm\sqrt{-5}$. In brief, he calls the former "infinity" and the latter a "complex" number.

The same principle is followed in dealing with the loci that correspond to the equations. A formal solution of the algebraic problem considered above is, under the conventions of coordinate geometry, also a formal solution of the problem of determining where the line $ax+by+c=0$ meets the conic $F=0$; and the postulate that there are always two actual solutions of the former is naturally followed by the postulate that there are two actual corresponding solutions of the latter. Thus the mathematician is led to assert that a line always meets a conic in two points, that sometimes the two are coincident, that sometimes one is or both are "at infinity," and that sometimes both are "imaginary."

Now, what is too little dwelt upon by teachers and text-book writers is that in the last sentence the word "point" is used in at least three widely different senses. Whatever the "points at infinity" and the "imaginary points" of a plane may be, they are certainly not points in the ordinary meaning of the term. Unfortunately the beginner rarely realises this. He gathers from his studies a hazy notion that there is upon the plane, beyond the uttermost horizon, an actual region called "infinity" where, in awful and mysterious seclusion, far-wandering lines and curves somehow fulfil their destiny; and that there is another region, under his eyes if he could but see it, where "imaginary" points enjoy their wholly inscrutable existence. And if his common sense resists these mystical ideas he comes to the conclusion either that geometry is beyond his mental grasp or else that it is "all rot."

Why do we teachers so often leave the matter in this unsatisfactory position? Partly, no doubt, because we feel the generalising instinct to be so essential in mathematics that it should be fostered even at the cost of some lack of clarity in thinking; but mainly, perhaps, because we were taught no better ourselves. The pioneers of the seventeenth and eighteenth centuries were too busy exploiting the field thrown open by the genius of Descartes to spend much time upon critical scrutiny of the instruments of their conquests. They were contented to speak metaphorically of "points" without understanding clearly what geometrical entities answered to the name and without relating them by definition to the ordinary, intuitable geometrical point. And our teachers held, too modestly, that what was good enough for the great men was good enough for them and for us. Truth to tell, satisfactory doctrines of imaginary points and of points at infinity are not easy to discover or to apprehend, and cannot even be enunciated here. The reader who needs a clear explanation of them must turn to the chapters on "complex space" and "ideal space" in such works as Professor E. H. Neville's admirable *Prolegomena to Analytical Geometry*. We can only insist that to think of a plane as containing points at infinity and imaginary points mixed up with the points of common sense is to conceive an entity as unmanageable as Strephon in *Iolanthe*, who, it will be remembered, was a fairy down to the waist and for the rest a mortal! Even Gilbert's ingenuity could not deal satisfyingly with such a being unless he turned him wholly into a mortal or wholly into a fairy. Similarly mathematics can get along quite happily with a plane composed of ordinary points or with a (metaphorical) plane composed of complex or of ideal points, but not with one in which the kinds are mixed.

The object of this article is to suggest an elementary treatment of asymptotes

which seeks to keep on the right side both of the critical modern mathematician and of the schoolboy (who is in his simpler way equally critical) by remaining strictly upon the level of common sense. That is, when a line really cuts a conic or curve of higher degree in only one point, we shall say so; and when it does not really cut the curve at all, again we shall say so. Whether the method is new or (as is much more probable) has also been thought of by others, the author cannot say; but it certainly seems to throw a very useful light upon asymptotes and the curves that possess them.

(1) We begin with the rectangular hyperbola $xy=a^2$ and note that while any line crossing the x-axis at an acute angle cuts the curve in two points, one on each branch, a line parallel to a coordinate axis cuts it in only one point, and that the axes themselves do not cut the curve at all. Let us consider the continuous band of parallel lines described by the equation $x=p$, where p starts from a positive value p_0 and diminishes continuously towards zero. Then it is evident (i) that all members of the band between $x=p_0$ and $x=0$ cut the curve in single points whose distance from the origin increases without limit as p approaches zero, and (ii) that the axis $x=0$ is the first member of the band which does not cut the curve. In technical language, the axis is a *limit* of this band of parallels which cut the hyperbola in one point only. Now it is proposed to base the idea of an asymptote upon the facts here exemplified. If $F=0$ is an equation of the nth degree, a line may cut the corresponding locus in a maximum of n points; but there may be one or more continuous bands of parallel lines which cut it in only m points, m being less than n. If such a band has as its limit a line cutting the curve in fewer than m points or not at all, and if as the parallels approach that limit one of the m points of intersection recedes endlessly far from the origin, then the limiting line is to be called an asymptote. In formal terms:

"An asymptote of a curve of the nth degree is the limit of a continuous series of parallel lines which cut the curve in less than n points and whose intersections therewith, as the lines approach the limit, become and remain farther from the origin than any given distance, however great."

It follows from the definition that the distance between the curve and an asymptote tends towards but never reaches zero. This is, of course, the property from which the asymptote derives its name. It is an odd result of the generalising tendency in mathematics that a Greek word implying that a certain line never meets a curve should acquire the meaning of meeting it twice!

Returning to the simple instance of the rectangular hyperbola, let us note that the facts pointed out can be derived analytically by substituting $x=p$ in the equation $xy=a^2$. The resulting equation $y=a^2/p$ shows that the line $x=p$ cuts the curve in only one point, that as p approaches zero the y-coordinate of the intersection becomes numerically great without limit, and that (since $a^2/0$ has no value) the axis $x=0$ does not cut the curve at all. According to the definition, then, the y-axis is an asymptote; and by a similar argument the x-axis is an asymptote also.

(2) For the sake of contrast consider next the parabola $y^2=4ax$. Here, while the maximum number of points in which a line may cut the curve is two, the line $y=p$ cuts it in a single point whose x-coordinate is $p^2/4a$. But since this expression yields a value for x for every value of p, the band of parallels that cut the hyperbola in one point only has no limiting line. That is, there is no asymptote.

(3) Next take the curve $y^2+axy+by+c=0$, for which Descartes gave, in his famous *Géometrie*, an ingenious mechanical construction.* Writing the equation in the form

$$y(y+ax)+by+c=0,$$

* See p. 52 of the translation by D. E. Smith and M. L. Latham (Open Court Series, 1925).

we observe that either of the substitutions $y=0$ or $y+ax=0$ reduces it from the second to the first degree, and that the same reduction must therefore follow from either of the substitutions $y=p$ or $y+ax=p$. The second substitution gives

$$x=(p^2+bp+c)/a(b+p),$$

from which we infer (i) that lines parallel to $y+ax=p$ cut the curve in only one point instead of in the two made possible by the degree of the equation, (ii) that x increases without limit as p approaches $-b$, and (iii) that there is no value of x when $p=-b$. It follows from the definition that $y+ax=-b$ is an asymptote.

The first substitution leads to

$$x=-(p^2+bp+c)/ap,$$

which has no value when $p=0$. Thus we infer that $y=0$ is also an asymptote.

(4) For the hyperbola $$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

that is $$\left(\frac{x}{a}+\frac{y}{b}\right)\left(\frac{x}{a}-\frac{y}{b}\right)=1,$$

the substitutions which reduce the equation from the second degree to the first are evidently $x/a \pm y/b=p$. From these we deduce $y=\pm b(1-p^2)2p$, which increases without limit as p approaches zero and for $p=0$ has no value. We conclude that the asymptotes are

$$x/a \pm y/b=0.$$

(5) In applying the method to curves of degree higher than the second, we need the following theorem. Let

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_{n-1}x+a_n=0$$

be an equation in which the value of a_0 depends and the values of the other coefficients *may* depend upon a variable parameter p, subject to the conditions (i) that when p approaches the value P, a_0 approaches zero continuously, and (ii) that when $|a_0|$ is less than A, $|a_1|$ is never less than B nor any of the coefficients numerically greater than C, A, B and C being specifiable numbers. Then as a_0 approaches zero one of the roots of the equation will increase without limit.

A general proof would require too long a digression. It is, however, easy to prove that the theorem holds good for the quadratic and cubic equations, which are the only ones needed in this article.

(i) If $ax^2+2bx+c=0$, we have $x=\{-b\pm\sqrt{b^2-ac}\}/a$; and since, as a tends to zero, b does not fall below B nor c rise above C, one root of the equation tends to the value $-2b/a$, which increases without limit.

(ii) It can be shown by substitution that the equation

$$ax^3+3bx^2+3cx+d=0$$

is satisfied (Cardan's solution) by

$$x=(-b+u^{\frac{1}{3}}+v^{\frac{1}{3}})/a,$$

where $$u=\tfrac{1}{2}(-G+\sqrt{G^2+4H^3}), \qquad v=\tfrac{1}{2}(-G-\sqrt{G^2+4H^3}),$$

$$G=-(u+v)=a^2d-3abc+2b^3, \quad H=-u^{\frac{1}{3}}v^{\frac{1}{3}}=ac-b^2.$$

Under the conditions laid down, when a tends to zero u and v both tend to the value $-b^3$. Thus one root tends to the value $-3b/a$, which increases without limit.

(6) To find the asymptotes of the cubic curve

$$x^3+x^2y-2xy^2-x^2+xy+4y^2-10=0,$$

i.e. $$x(x-y)(x+2y)-x^2+xy+4y^2-10=0,$$

we note that the degree of the equation will be reduced from the third to the second by any one of the three substitutions $x=p$, $x-y=p$, $x+2y=p$. Thus the three bands of parallels given by these linear equations will all cut the curve in two points instead of the three indicated by the degree of the curve. If, then, any of them contains a limiting line which cuts the curve in only one point or not at all, the line may be an asymptote.

The substitution $x=p$ reduces the equation to the quadratic

$$2(2-p)y^2+p(p+1)y-(10+p^2-p^3)=0.$$

When $p=2$ the equation suffers a further reduction to the first degree, *i.e.* the line $x=2$ cuts the curve in only one point, namely $(+2, +1)$. Also, by the theorem of (5), as p approaches 2 one value of y given by the quadratic increases without limit—that is, one of the two intersections of $x=p$ with the cubic recedes to a limitless distance from the origin. We conclude, then, that $x=2$ is an asymptote.

Making next the substitution $x-y=p$, we obtain

$$(3p+4)y^2+p(4p-1)y-(10+p^2-p^3)=0,$$

which reduces to the first degree when $p=-\frac{4}{3}$. Whence we deduce by the same reasoning as before that $x-y=-\frac{4}{3}$ is an asymptote.

Finally the substitution $x+2y=p$ gives

$$2(3p-1)y^2+5p(p-1)y-(10+p^2-p^3)=0;$$

whence it follows that there is a third asymptote, $x+2y=\frac{1}{3}$.

(7) The equation of the fourth degree

$$x^4-xy^3+3y^2-x-1=0,$$

i.e.
$$x(x-y)(x^2+xy+y^2)+3y^2-x-1=0,$$

is reduced to the third degree by putting $x=p$ or $x-y=p$. The first substitution gives

$$-py^3+3y^2+p^4-p-1=0,$$

and the second gives

$$3py^3+3(2p^2+1)y^2+(4p^3-1)y+p^4-p-1=0.$$

These equations show that the bands of lines $x=p$ and $x-y=p$ cut the curve in three points only, and that $x=0$ and $x-y=0$ are asymptotes cutting the curve in two points.

(8) If the equation of a curve is given in, or can be thrown into the form exemplified by

$$(x+y-2)(2x-y+5)(2x+3y-4)=3x-y+3,$$

the asymptotes can be inferred immediately. For the cubic is reduced by the substitution $x+y-2=p$ to a quadratic in which the coefficient of the highest term approaches zero with p. This is further reduced to a linear equation when p receives the value zero. Thus $x+y-2=0$ must be an asymptote. By the same reasoning $2x-y+5=0$ and $2x+3y-4=0$ are also asymptotes.

The asymptotes of $\quad x^2y^2-x^2y-xy^2+4x+4y+1=0$

could be determined in this way. For the equation can be thrown into the form

$$xy(x-1)(y-1)=xy-4x-4y-1.$$

By equating any factors on the left-hand side to p we reduce the equation to a quadratic, in which the coefficient of the highest term involves p, and by making p zero reduce it further to a linear equation. Hence the asymptotes are $x=0$, $y=0$, $x=1$, $y=1$.

(9) A curve with an equation such as

$$(x-y)(x+2y-1)(x+2y+4)=7x-4y-10,$$

in which two of the factors on the left have the same variable terms, presents features of special interest. As in (8) we see that the equation $x-y=p$ describes a band of parallels whose ordinary members cut the cubic in two points, while the special line $x-y=0$ cuts it in only one point and is an asymptote. But when we consider the other factors we note that the substitution $x+2y=p$ reduces the equation to linearity. That is, all lines parallel to $x+2y=p$ cut the curve in one point only, so that an asymptotic line, if there be one, must not cut the curve at all. The values of p which yield such lines may be determined in our usual way by actual substitution, which reduces the equation to

$$3(p-2)(p+5)y=p(p-1)(p+4)-7p+10.$$

From this we infer that the asymptotes are $x+2y=2$ and $x+2y=-5$.

But there is a neater and more instructive way of proceeding.* The original equation can be thrown into the form

$$(x-y)(x+2y-2)(x+2y+5)=x+2y-10,$$

in which the variable terms repeated on the left now appear also on the right. As before, the substitution $x+2y=p$ will in general reduce the equation to linearity, but it is evident that neither of the lines $x+2y=2$ and $x+2y=-5$ can cut the curve at all. For they must cut it where they cut the line $x+2y=10$ to which they are actually parallel. They are, then, a pair of parallel asymptotes.

Similarly, since the equation

$$(x+y)(2x-3y)^2+2(x+y)(2x-3y)=7x-3y+9$$

can be thrown into the form

$$(x+y)(2x-3y-1)(2x-3y+3)=2(2x-3y)+9,$$

we may infer that the curve has two parallel asymptotes $2x-3y-1=0$ and $2x-3y+3=0$ in addition to $x+y=0$, the latter being associated with a band of lines cutting the curve in two points, the two former with a band of lines that cut it in single points. In the case of

$$(x+y)(2x-3y)^2+2(x+y)(2x-3y)=x-4y+5,$$

i.e. $$(x+y)\{(2x-3y)^2+2(2x-3y)+1\}=2x-3y+5,$$

the two parallel asymptotes reduce to a single one, $2x-3y=-1$; while

$$(x+y)(2x-3y)^2+2(x+y)(2x-3y)=x-9y-1,$$

i.e. $$(x+y)\{(2x-3y)^2+2(2x-3y)+3\}=2(2x-3y)-1,$$

has no parallel asymptotes, since the quadratic factor on the left cannot be split into linear factors.

The last example suggests that the equations in which the terms of highest degree contain a repeated linear factor from which we can deduce the existence of parallel asymptotes are best regarded as special cases of the general form

$$(a_1x+b_1y+c_1)\{(a_2x+b_2y)^2+ux+vy\}=Ax+By+C.$$

The second factor on the left-hand side, when equated to a variable p, represents a family of parabolas which cut the cubic in only two instead of the maximum of six points, and has a limiting member which cuts it only in one point and may be called an asymptotic parabola. For instance, the equation

$$x(x+y)^2-3x^2-xy+2y+2=0$$

can be thrown into the form

$$x\{(x+y)^2-3x-y-2\}=-2(x+y)-2.$$

* I owe the rest of (9) to Professor Neville, to whom I am indebted for helpful criticisms of the first draft of this article.

Now a point (x, y) which is on both the cubic and the parabola

$$(x+y)^2-3x-y-2=p$$

must satisfy both the latter equation and the equation $px=-2(x+y)-2$ to which the cubic is reduced by substitution; that is, the parabola cuts the cubic in the two points where it cuts the line $px=-2(x+y)-2$. But if $p=0$, this line becomes $x+y+1=0$, which is parallel to the axis of the parabola, and therefore cuts it in only one point. The substitution of a variable k for the coefficient 3 in the equation of the cubic would cover the existence of an endless number of families of parabolas $(x+y)^2-kx-y+2=p$, all having the properties described. But if $k=1$ the parabola degenerates into the pair of parallel lines $x+y=2$ and $x+y=-1$. Thus a pair of parallel asymptotes may be regarded as a degenerate asymptotic parabola.

(11) In conclusion, we note that if a curve is of the nth degree the sum of the terms of that degree in its equation contains not more than n linear factors. And since each of these may, in turn, be equated to p, the degree of the equation may be reduced in not more than n different ways. Since, further, each of these reductions may lead to the determination of an asymptote, it follows that a curve of the nth degree may possess n asymptotes, but no more.

T. P. NUNN.

GLEANINGS FAR AND NEAR.

354. The very excellent Gilbert White, of Selborne, one of the very few authors who should be cited by the title of *His Readableness*, has an amusing mistake as follows :—

He thinks the long-legged plover must be, for his weight, a much longer bird than the flamingo ; and thus he makes it out : the latter weighs sixty-four ounces, the former four and a quarter. But the plover has eight inches length of leg ; if, then, the flamingo were as long-legged in proportion to his weight, its leg would be more than ten feet in length, whereas it is only twenty inches. Consequently the plover is out of all comparison a longer-legged bird than the flamingo.

Now Mr. White had an idea that the weight of a system increases in the same proportion as its linear dimensions, and no one of his editors has set him right. But had he made the legs grow, as he ought to have done, in the proportion of the cube root of the weights, he would have given his flamingo a leg of nineteen inches and three quarters—very close to the truth.

Therefore, people should not write, etc., etc., without etc., etc. Q.E.D. —*Athenaeum*, 1845, p. 1180.

[From internal evidence, in the letter from which this is extracted, Q.E.D. is De Morgan. The letter referred to by De M. was written to Barrington, May 7, 1779.]

355. Nobody has yet either in England or France come near in the scale of artificial verse or prose to the excellent bishop in the eighth century who devoted some months to composing thirty-five verses of prayer for Charlemagne, which when read perpendicularly, horizontally, and along the lines of an inscribed rhomboid, gave eight other acrostic verses to the same effect. —Morley, *Recollections*, ii. p. 95.

356. Anyone who has done no more than the first book of Euclid's geometry ought to have got into his head the notion of a demonstration of the rigorously close connection between a conclusion and its premisses, of the necessity of being able to show how each link in the chain comes to be where it is, and that it has a right to be there. This, however, is a long way from the facts of real life, and a man might well be a great geometer, and still be a thoroughly bad reasoner in practical questions.—Morley, *Popular Culture*.

HOW *e* IS TO BE INTRODUCED INTO OUR TEACHING.

BY W. MILLER, D.Sc., Ph.D.

THE following solution is suggested of Mr. J. Katz's problem, "How is *e* to be introduced into our teaching?" published in the December number of the *Gazette*.

The students with a little team work should verify the following table:

Number.	2	3	6	10	·3
Sq. rt.	1·414214	1·732051	2·449490	3·162277	1 − ·452277
4th ,,	1·189207	1·316074	1·565085	1·778280	1 − ·259917
8th ,,	1·090507	1·147203	1·251033	1·333522	1 − ·139719
16th ,,	1·044274	1·071075	1·118496	1·154782	1 − ·072487
32th ,,	1·021897	1·034928	1·057590	1·074608	1 − ·036925
64th ,,	1·010889	1·017314	1·028392	1·036633	1 − ·018636
128th ,,	1·005430	1·008620	1·014097	1·018152	1 − ·009362
256th ,,	1·002711	1·004301	1·007024	1·009035	1 − ·004692
512th ,,	1·001355	1·002148	1·003506	1·004507	1 − ·002349
1024th ,,	1·000677	1·001074	1·001751	1·002251	1 − ·001175

The preliminary discussion of this table might be dispensed with, but Mr. Katz postulates "that the new matter presented to the pupil must be a natural and inevitable development from what he knows already."

From the above table the student sees:

(1) That the successive square roots of all numbers tend to 1.

(2) That the difference from 1 tends to be double the difference in the succeeding square root. The digits of the difference in respect of which this statement is true are printed in italics. The numbers composed of italicised digits will be referred to as the *logarithms* of the number at the head of the column.

(3) Logarithms of different numbers may be said to belong to the same *order* if they are derived from the numbers by the same number of root extractions.

(4) Logarithms (of the same order) of different numbers have for sum the logarithm (of the same order) of their product.

Thus for the numbers 2, 3, 6, we have

$$\cdot 000677 + \cdot 001074 = \cdot 001751, \quad \text{I.}$$

these three logarithms appearing side by side in the table.

Also for numbers ·3, 10, 3, we have

$$-\cdot 001175 + \cdot 002251 \doteqdot \cdot 001074. \quad \text{II.}$$

(5) Logarithms of numbers less than 1 are *negative*.

(6) The logarithm of 1 must always be *zero*.

(7) Equations I. and II. are unaltered by multiplication by a constant factor k. We might, then, define a *system* of logarithms as the logarithms of any order multiplied by a constant factor.

(8) The higher the order from which the system is derived the more accurate the system.

If N is *any* number, the system is derived from $k(\sqrt[m]{N}-1)$ by comparison of the digits with those of the order higher. Here m is a power of 2, viz. 2^n, and the higher order is represented by 2^{n+1}.

(9) The *ideal* logarithm would be of an infinitely high order, but would be infinitely small unless the factor k were infinitely large.

(10) The *natural* factor for a system of order 2^n would be the factor 2^n. There would then be agreement of results of all orders.

(11) The *natural* logarithm of 10 would be $2^{10}\times\cdot002251$ or approximately $2\cdot30\ldots$.

(12) Natural logarithms are not the most convenient in practice. It is more convenient to have 1 as the logarithm of 10 than $2\cdot30\ldots$, *i.e.* it would be of practical convenience to make our factor k equal to $\dfrac{1}{2\cdot30\ldots}$, *i.e.* the factor k would be the transition factor from *natural* logs. to practical logs.

(13) The idea embodied in (12) is to derive our practical logarithms from the definition $\dfrac{\sqrt[m]{N}-1}{\sqrt[m]{10}-1}$, where $m=2^n$, and in the ideal case n becomes infinite.

10 is then said to be the *base* of the system.

More generally, any base a (not unity) may be taken, and

$$\log_a N=\operatorname*{Lt}_{m\to\infty}\frac{\sqrt[m]{N}-1}{\sqrt[m]{a}-1},\quad\text{where } m=2^n.$$

(14) If natural logarithms have a base, what is it? The natural logarithm is derived from $m(\sqrt[m]{N}-1)$. By comparison we have

$$m=\frac{1}{\sqrt[m]{a}-1}\quad\text{or}\quad 1+\frac{1}{m}=\sqrt[m]{a}\quad\text{or}\quad a=\left(1+\frac{1}{m}\right)^m.$$

The base of natural logarithms, generally denoted by e, is $\left(1+\dfrac{1}{m}\right)^m$, where $m=2^n$ and n is an infinitely large integer.

(15) We may write $T_n=\dfrac{1}{m}\,\dfrac{\sqrt[m]{N}+1}{\sqrt[m]{N}-1}$, where $m=2^n$.

Since $\sqrt[m]{N}+1\to 2$ as n becomes infinite, it is easily seen that $T_\infty=\dfrac{2}{\log_e N}$.

(16) The method of obtaining logarithms by extracting a large number of square roots is long and laborious. This labour would be shortened by directly deriving one from another, $T_0, T_1, T_2, \ldots T_\infty$.

The necessary relation is suggested by the identity

$$\frac{\sqrt{x}+1}{\sqrt{x}-1}+\frac{\sqrt{x}-1}{\sqrt{x}+1}\equiv 2\,.\,\frac{x+1}{x-1},$$

which may be written

$$\frac{1}{2^n}\cdot\frac{\sqrt{x}+1}{\sqrt{x}-1}+\frac{\dfrac{1}{4^n}}{\dfrac{1}{2^n}\,\dfrac{\sqrt{x}+1}{\sqrt{x}-1}}\equiv\frac{1}{2^{n-1}}\cdot\frac{x+1}{x-1}.$$

Now write $\sqrt[m]{N}$ for x, where $m=2^{n-1}$.

Then $$T_n+\frac{1}{4^nT_n}\equiv T_{n-1}\quad\text{or}\quad T_n^2-T_{n-1}\,.\,T_n+\frac{1}{4^n}=0.$$

If we know T_{n-1}, we can calculate T_n by solving a quadratic equation. As there are various ways of rapidly arriving at T_∞, and therefore $\log_e N$, we shall select some of the simplest. It should be noted that

$$T_0=\frac{N+1}{N-1}.$$

(17) From the definitions it is clear that $\log_a a=1=\log_e e$.

(18) The crude observation of the additive properties of logarithms under (4) is insufficient.

In the definition of $\log_a N$ put $\dfrac{1}{M}$ for N. Then

$$\log_a \frac{1}{M} = \underset{m\to\infty}{\mathrm{Lt}} - \frac{1}{\sqrt[m]{M}} \frac{\sqrt[m]{M}-1}{\sqrt[m]{a}-1} = - \underset{m\to\infty}{\mathrm{Lt}} \frac{\sqrt[m]{M}-1}{\sqrt[m]{a}-1} = -\log_a M.$$

In the identity $$\frac{x-1}{x+1}+\frac{y-1}{y+1} \equiv (xy-1)\frac{2}{(x+1)(y+1)}$$

put $x=\sqrt[m]{N}$, $y=\sqrt[m]{M}$, multiply across by $\dfrac{2}{\sqrt[m]{a}-1}$, and let $m\to\infty$.

It follows directly that

$$\log_a N + \log_a M = \log_a MN.$$

Similarly it follows that

$$\log_a \frac{N}{M} = \log_a N + \log_a \frac{1}{M} = \log_a N - \log_a M.$$

(19) It will be observed that no *general* theory of indices has been assumed, the inverted concept of negative and fractional indices in the traditional treatment being almost as puzzling to the average pupil as e.

We define $N^{p/q}$ as $\sqrt[q]{N^p}$, where p and q are positive integers and $N^{-p/q}$ as $\dfrac{1}{N^{p/q}}$.

It follows easily from (18) that

$$p\log_a N = \log_a N^p = \log_a (N^{p/q})^q = q\log_a N^{p/q}.$$

Hence $$\log_a N^{p/q} = p/q \log_a N.$$

Also $$\log_a N^{-p/q} = \log_a \frac{1}{N^{p/q}} = -\log_a N^{p/q}, \text{ by (18)},$$
$$= -p/q \log_a N.$$

Hence generally $$\log_a N^n = n\log_a N,$$

and since $\log_a a = 1$, $$\log_a a^n = n.$$

Logarithms and indices are thereby identified.

Also $$\log_a \frac{M^m N^n}{S^s} = m\log_a M + n\log_a N - s\log_a S.$$

(20) Let us return now to the calculation of logarithms from

$$T_n^2 - T_{n-1}\,.\,T_n + \frac{1}{4^n} = 0, \quad T_0 = \frac{N+1}{N-1}, \quad \log_e N = \frac{2}{T_\infty}.$$

The immature mathematician prefers numerical cases to generalities. We select $N=\frac{4}{3}$ because we see from the table at the beginning that with numbers near 1 the logarithms develop out more rapidly.

$$T_0 = 7 \quad\text{and}\quad T_1^2 - 7T_1 + \tfrac{1}{4} = 0.$$

The sum of the roots is 7, so that, if T_1 is one root, $7-T_1$ is the other.

$$T_1 = 3{\cdot}5 + 2\sqrt{3}, \quad 7-T_1 = 3{\cdot}5 - 2\sqrt{3},$$

i.e. $$T_1 = 6{\cdot}9641, \quad 7 - T_1 = {\cdot}0359.$$

It is observed that T_1 differs little from T_0. This is in the nature of *natural* logarithms, as was pointed out in (10). There is no need, as will be shown immediately, for solving a second quadratic equation except to get our bearings *once* with regard to accuracy.

The second equation is $T_2^2 - 6{\cdot}9641T_2 + \frac{1}{16} = 0$, whence $T_2 \doteqdot 6{\cdot}955$,

These two equations may be written

$$T_0 - T_1 = \frac{1}{4 \times 6{\cdot}9641}, \text{ or approx. } {\cdot}04,$$

$$4(T_1 - T_2) = \frac{1}{4 \times 6{\cdot}955}.$$

From which we easily conclude that

$$T_0 - T_1 = 4(T_1 - T_2)$$

to an error in the fourth significant figure, *i.e.* in the *fifth decimal place.*

Clearly this will hold for succeeding equations.

Hence
$$\begin{aligned} T_0 - T_1 &= 4(T_1 - T_2), \\ T_1 - T_2 &= 4(T_2 - T_3), \\ T_2 - T_3 &= 4(T_3 - T_4), \\ &\dots\dots\dots\dots \\ \dots - T_\infty &= 4(\dots - T_\infty). \end{aligned}$$

Adding
$$\begin{aligned} T_0 - T_\infty &= 4(T_1 - T_\infty); \\ \therefore\ 3T_\infty &= 3T_1 - (T_0 - T_1); \\ \therefore\ T_\infty &= T_1 - \tfrac{1}{3}(T_0 - T_1) \\ &= \text{larger root} - \tfrac{1}{3} \text{ smaller root of the first quadratic equation.} \end{aligned}$$

Hence
$$T_\infty = 6{\cdot}9641 - {\cdot}0120 = 6{\cdot}9521$$

and
$$\log_e \tfrac{4}{3} = \frac{2}{6{\cdot}9641} = {\cdot}28768.$$

Hence any number from 1 to $\frac{4}{3}$ can have its logarithm calculated with this accuracy at least by the solution of *one* quadratic equation.

(21) A simple formula may be derived from

$$T_\infty = T_1 - \tfrac{1}{3}(T_0 - T_1) \quad \dots\dots\dots\dots\text{(A)}$$

and
$$T_0 - T_1 = \frac{1}{4T_1}. \quad \dots\dots\dots\dots\text{(B)}$$

Substituting from (B) in (A),

$$T_\infty = T_1 - \frac{1}{12T_1}. \quad \dots\dots\dots\dots\text{(C)}$$

In the denominator of (B) and (C) we may put T_0 for T_1, so that

$$T_1 \doteqdot T_0 - \frac{1}{4T_0}.$$

Substituting in (C),

$$T_\infty = T_0 - \frac{1}{4T_0} - \frac{1}{12T_0} = T_0 - \frac{1}{3T_0}.$$

Hence
$$\frac{2}{\log_e N} = \frac{3T_0^2 - 1}{3T_0} \quad \text{or} \quad \log_e N = \frac{6T_0}{3T_0^2 - 1}.$$

(Problem : Calculate e from $\log \sqrt[4]{e} = \frac{1}{4}$.)

(22) We may now proceed with the calculation of logarithmic tables by either method.

For $N = \frac{5}{4}$, $T_0 = 9$, and

$$\log_e N = \frac{54}{242} = \frac{27}{11 . 11} = 2{\cdot}454545 \div 11$$
$$= {\cdot}22314.$$

Similarly, $\log_e \frac{6}{5} = \frac{33}{181} = \cdot 18232,$

$$\log_e 10 = \log_e (\tfrac{4}{3})^3 (\tfrac{5}{4})^4 (\tfrac{6}{5})^3 = \underline{2 \cdot 30256.}$$

This number is really more important than e.

The logarithms found to the base e can now be converted to the base 10.

The logarithms of 2, 3, 4, ... 9 can be calculated from the scheme:

$$N = (\tfrac{4}{3})^x (\tfrac{5}{4})^y (\tfrac{6}{5})^z$$

N	x	y	z
2	1	1	1
3	1	2	2
4	2	2	2
5	2	3	2
6	2	3	3
8	3	3	3
9	2	4	4

The logarithm of any number can now be found by ratios. Let the number be 1·4823. An easily found approximation is $\frac{3}{2}$ or 1·5. Calculate $\log_e \dfrac{1 \cdot 5}{1 \cdot 4823}$, for which $T_0 = \dfrac{2 \cdot 9823}{\cdot 0177}$ by one quadratic equation, and deduce the required log.

Prime numbers are similarly treated. Thus

$$\log_e \tfrac{7}{6} = \tfrac{39}{253} = \cdot 15415.$$

Hence $\log_e 7$.

Note that $\log_e (\frac{7}{6})^2 = \cdot 30830$

and $\log_e 2 = \cdot 69314.$

Hence $\log_e \frac{49}{18} = 1 \cdot 00144.$

Hence e must be nearly $\frac{49}{18}$.

Having calculated log. tables we are justified in finding e from the tables, but an approximate value will be found later by the principle of proportional parts.

(23) Since $T_0 = \dfrac{N+1}{N-1}$, T_0 is large when N approaches 1.

The nearer N is to 1 the more nearly is

$$\log_e N = \frac{6T_0}{3T_0^2 - 1}.$$

Suppose δx small in $\log\left(1 + \dfrac{\delta x}{x}\right)$, then $T_0 = \dfrac{2x + \delta x}{\delta x}$ and is large. When δx is small enough we may neglect 1 in comparison with $3T_0^2$.

Hence $$\log_e (x + \delta x) - \log_e x \to \frac{2}{T_0} \quad \text{or} \quad \frac{\delta x}{x}.$$

The principle of proportional parts now follows in the usual way.

We may obtain from this relation an approximation to e.

$$\log_e \tfrac{49}{18} - \log_e e = \cdot 00144;$$

$$\therefore \ \delta x = \cdot 00144 \times \tfrac{49}{18} = \cdot 00392;$$

$$\therefore \ e = \tfrac{49}{18} - \cdot 00392 = 2 \cdot 7183.$$

(24) From $\log_e (x + \delta x) - \log_e x \to \delta x / x$ as $\delta x \to 0$ we have

$$\frac{d}{dx} \log_e x = \frac{1}{x} \quad \text{...I.}$$

or $$\log_e x = \int \frac{1}{x}\, dx,$$

and we are now in a position to square the hyperbola.

In equation I. let $\log_e x = \xi$, then from the identity of indices and logarithms $x = e^{\xi}$.

Hence equation I. becomes

$$\frac{d\xi}{de^{\xi}} = \frac{1}{e^{\xi}} \quad \text{or} \quad \frac{de^{\xi}}{d\xi} = e^{\xi}.$$

By assuming $\qquad e^{\xi} = a_0 + a_1\xi + a_2\xi^2 + \dots + a_n\xi^n + \dots$

and using $e^0 = 1 = a_0$, we may find the expansions for e^{ξ} and e.

(25) The more advanced student of mathematics may demand greater rigour in the theory than has been set out above.

The definition of T_n shows that $T_n \gtrless 0$ as $N \gtrless 1$.

Our fundamental equation may be written

$$T_{n-1} = T_n + \frac{1}{4^n T_n}, \quad \text{showing that } |T_{n-1}| > |T_n|,$$

$$\sum_1^{\infty} T_{n-1} = \sum_1^{\infty} T_n + \sum_1^{\infty} \frac{1}{4^n T_n}.$$

Hence, as $N \gtrless 1$, $\qquad T_0 - T_\infty \gtrless \frac{1}{T_0} \sum_1^{\infty} \frac{1}{4^n},$

$$\text{i.e.} \quad T_0 - T_\infty \gtrless \frac{1}{3T_0},$$

$$\text{i.e.} \quad T_\infty \lessgtr T_0 - \frac{1}{3T_0}. \quad \dots\dots\dots\dots (1)$$

Also as $N \gtrless 1$, $\qquad T_0 - T_\infty \lessgtr \frac{1}{T_\infty} \sum_1^{\infty} \frac{1}{4^n}, \quad \text{i.e.} \lessgtr \frac{1}{3T_\infty};$

$$\therefore\ 3T_\infty^2 - 3T_0 T_\infty + 1 \mp \epsilon = 0, \quad \text{as } N \gtrless 1,$$

ϵ being a positive quantity.

$$\therefore\ T_\infty = \frac{3T_0 \pm \sqrt{9T_0^2 - 12 \pm 12\epsilon}}{6};$$

$$\therefore\ T_\infty \gtrless \frac{T_0}{2}\left(1 + \sqrt{1 - \frac{4}{3T_0^2}}\right) \quad \text{as} \quad N \gtrless 1,$$

$$\text{i.e.} \quad \log_e N \lessgtr \frac{4}{T_0\left(1 + \sqrt{1 - \frac{4}{3T_0^2}}\right)} \quad \text{as} \quad N \gtrless 1. \quad \dots\dots\dots (2)$$

Now put $1 + x$ for N and $\frac{2+x}{x}$ for T_0 in (1) and (2). Then, according as $x \gtrless 0$,

$$\frac{6x + 3x^2}{6 + 6x + x^2} \lessgtr \log_e(1+x) \lessgtr \frac{4x}{(2+x)\left(1 + \sqrt{1 - \frac{4x^2}{3(2+x)^2}}\right)}.$$

From which it is clear that

$$\log_e(1+x) \to x \quad \text{as} \quad x \to 0.$$

Hence we may write $\qquad \log_e\left(1 + \frac{\delta x}{x}\right) = \frac{\delta x}{x} + \epsilon(\delta x)^2,$

where ϵ is finite.

Multiplying by $\frac{x}{\delta x}$, we get

$$\log_e\left(1 + \frac{\delta x}{x}\right)^{x/\delta x} = 1 + \epsilon x\, \delta x.$$

x may be any finite number, and δx is small.

Hence $$\underset{m\to\infty}{\text{Lt}} \log_e\left(1+\frac{1}{m}\right)^m = 1 = \log_e e.$$

Hence $\underset{m\to\infty}{\text{Lt}} \left(1+\frac{1}{m}\right)^m = e$ without restrictions on m.

Also, since $\log_e(1+\delta x)\to\delta x$ as $\delta x\to 0$,

$$1+\delta x\to e^{\delta x} \quad \text{or} \quad \delta x\to e^{\delta x}-1.$$

If $x=\log_e N$, $$e^x = N \quad \text{and} \quad e = N^{1/x}.$$

Hence $$x\to\frac{x}{\delta x}(N^{\delta x/x}-1),$$

i.e. as $m\to\infty$, $$\log_e N\to m(\sqrt[m]{N}-1),$$

and the restrictions on m in the definition of $\log_e N$ are removed.

The above is taken from manuscript on the "Arithmetical Fundamentals of Mathematics," in which circular and hyperbolic functions and elliptic integrals and functions are calculated to any accuracy by similar *ab initio* arithmetical methods. New types of arithmetical operations are also developed capable of solving any algebraic equation without algebraic theory.

WILLIAM MILLER.

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357. The ignorance of the clergy of (Roger Bacon's) time as to mathematics or physics was afterwards described by Anthony à Wood, who says that they knew no property of the circle except that of keeping out the devil, and thought that the points of a triangle would wound religion.—De Morgan, "Roger Bacon," *Penny Cyclopaedia.*

358. Mr. *William Oughtred*, a native Scholar and Fellow of *Eaton*, bred in *Kings-colledge Cambridge*, and (his Mathematical Studies (wherein by Study and Travel he so excelled, that the choicest Mathematicians of our age own much of their skill to him, whose house was full of young * Gentlemen, that came from all parts to be instructed by him) leading him to a retired and abstracted life) preferred onely by *Thomas* Earl of *Arundel* to *Albury* in *Surrey*, where having a strong persuasion upon principles of Art (much confirmed by the Scheme of his Majesties return in 1660. Sent his Majesty some years before by the Bishop of *Avignon*) that he should see the King restored; he saw it to his incredible joy and had his *Dimittis* a month after, *June* 30. 1660. and the 86. year of his age. Much requested to have lived in *Italy*, *France*, *Holland*, when he was little observed in *England*; as facetious in Greek and Latine, as solid in Arithmetique, Astronomy, and the sphere of all Measures,† Musick &c. exact in his stile, as in his judgement, handling his Cube [?Tube], and other Instruments at eighty, as steadily, as others did at thirty; owning this, he said, to temperance and Archery, principling his people with plain and solid truths, as he did the world with great and useful Arts, advancing new Inventions in all things but Religion. Which in its old order and decency, he maintained secure in his privacy, prudence, ‡ meekness, simplicity, resolution, patience, and contentment.—*Memoires of the Lives, Actions, Sufferings and Deaths of those Noble, Reverend and Excellent Personages that suffered by Death, Sequestration, Decimation, or otherwise, for the Protestant Religion, And the great Principle thereof, Allegiance to their Soveraigne, etc. etc. etc. etc.* 1668 (*per* Mr. A. T. Wicks).

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* *As Sir* William Backehouse son, *Mr.* Stokes, *Dr.* Will. Lloyd, *Mr.* Arch. Haughton, *who had much ado to prevail with his modesty to publish his* Trigonometria.

† *In the Mathematical way.*

‡ *See more of him in Learned and ingenious Mr.* Wase *his* 'Εγκωμιαστικόν (?...) *him before his* Trigonometria [word in brackets indecipherable].

THE APPROACH TO THE DIFFERENTIATION AND INTEGRATION OF LOGARITHMIC AND EXPONENTIAL FUNCTIONS.*

By J. M. Child, M.A.

In teaching beginners the elements of the calculus, the teacher's first real difficulty arises when he has to present the differentiation of the logarithmic and exponential functions; these being a natural outcome of an attempt to complete the rule for x^n for all values of n, including the case when $n=-1$. He either has to make the approach as in texts on "practical mathematics," by differentiating the series for e^x, with all the tacit assumptions that have to be made; or he is dependent on non-rigorous treatment of limits, necessary to show that $(a^x-1)/x$ tends to the limiting value $\log_e a$; or to find solutions of $dy/dx=y$ by some such process as is given in Lamb's treatise on the Calculus, without the necessary proofs of convergence and differentiability of the series assumed; or he is bound to discuss convergence and continuity fairly fully, and give such proofs as Lamb gives, or the equally severe method of Hardy's text.

To those who think that the latter alternatives are entirely unsuitable to beginners, and yet have understandable qualms about the non-rigorous presentment, it may be interesting to consider the following suggestions, which make no very great demands on algebraical knowledge, follow the line of rigorous treatment approximately, and lead directly to the Hardy treatment of the functions.

I suppose, of course, that the beginner has considered the infinite geometrical progression, and that the "sum" of such a series has been established as the limit of a sequence of which the nth member is the sum to n terms of the series. Also that this had led to some consideration of the existence of the limit of monotone sequences—one of the so-called intuitional proofs being sufficient at this stage.

I suppose also that the course has been one following some of the excellent elementary texts, in which integration has been introduced right at the beginning and worked at collaterally with differentiation:

(*a*) Growth of function; average rate of growth, limit. The whole being carried out arithmetically at first, with immediate applications.

(*b*) Differentiation of $u+A$, Au, $u+v$, $u-v$; illustrated arithmetically, geometrically and graphically, introducing "gradient" and "tangent."

(*c*) Area under graph as limit of either inferior or superior rectangles; proof of existence for equidistant ordinates whose number is of form 2^n+1. This area, A, is a function of x, and dA/dx is the last ordinate; hence, integration is inverse of differentiation.

(*d*) Indefinite integral introduced as a definite integral in which the upper limit is x, and the lower an arbitrary constant A, or, preferably the lower limit omitted and supposed to be some constant, real or imagined, which makes the integral function zero.

* The following article was written before I saw the suggestions of Mr. J. Katz in the December issue of the *Mathematical Gazette*. With this article I, frankly, do not agree. The approach is on wrong lines: the connection of logarithms with the hyperbola is quite a secondary consideration; the use of the hyperbola to obtain an arithmetical approximation to e, without any further knowledge of it, is not conducive to a correct idea of the exponential; and the fact that e is $\lim (1+1/n)^n$ is immaterial to the calculus course.

What is required is (i) the direct association of logarithmic and exponential functions; (ii) some attempt to show their continuity, the method being used which is suitable to the particular class taught, (iii) the demonstration that e is some number which can be used as a base of a set of logarithms; (iv) the evaluation of the integrals of a^x and $1/x$ in terms of exponentials and logarithms.

In this way the beginner is from the first convinced that the integral is a function of the *limits* and that, say,

$$\int^{x} x^n dx = \int^{x} u^n du = \int^{x} v^n dv = x^{n+1}/(n+1) - K,$$

where K is some constant depending on the bottom arbitrary constant limit, and therefore itself arbitrary; and that K is supposed to be zero when the integral is taken as an indefinite integral, which is then considered as a function of the top limit only.

With these suppositions, the suggested approach to the consideration of exponentials and logarithms, which in my opinion should most certainly be considered together, is as follows:

1. Plot 10^x from statistics obtained as follows: find the square root of 10, the square root of this square root, and so on, as far as is desired; by multiplication of these roots obtain the values of y for $x=\frac{3}{4}$, etc.; the reciprocals of these give values of y for negative values of x. Remark on the general form of the graph, always positive, single-valued, and so on. Interchange the axes, replotting the graph, and show why the graph then represents $y=\log_{10}x$; verify by means of logarithmic tables. Bring out the fact that the interchange of axes can be brought about by a rotation through a positive right angle and a reflection, thus avoiding the necessity of replotting the graph; this idea being common to all inverse functions, such as inverse circular functions and so on.

2. Plot a^x for several values of a. This can be readily done by finding powers of a geometrically. Note that the general forms of all these graphs is that of the graph of 10^x, and that any one of them can be "stretched" parallel to one or other of the coordinate axes to get any other. Note that they all pass through the point (0, 1). Convert by interchange of axes into the logarithmic curves for different values of the base, and note that these now all pass through the point (1, 0).

3. The gradient is the limit of the average gradient; from the graph it seems reasonable to assume that this exists. For each exponential graph in turn draw pairs of equidistant ordinates, the distance between each pair being some length k; draw the chord through the extremities of each pair and produce to meet the axis of x; verify that the part intercepted on the axis between the chord and, say, the left-hand ordinate is the same for each pair; that this length is different for different values of k for the same curve, and different for different curves for the same value of k. Verify a similar thing with regard to the logarithmic curves and ordinates to the axis of y. Finally consider the tangents and their gradients and bring out the constancy of the sub-tangents, on the axis of x in one case, and on the axis of y in the other; and show how this gives a rapid method of drawing exponential and logarithmic curves to a high degree of approximate accuracy, by the use of squared paper. Observe that the angle of slope of the exponential curve is the complement of the angle of slope of the corresponding logarithmic curve for the same values of x and y; *i.e.* for a curve considered as $y=a^x$ and $x=\log_a y$ in turn, dy/dx is the reciprocal of dx/dy. Also observe that the slope of the curve can apparently be any angle whatever at the point (0, 1) for the exponential curve; that is to say, it is feasible to assume that there is one curve for which $dy/dx=1$ at the point (0, 1).

With this preliminary graphical consideration the beginner should be in a position to follow and appreciate all the points of the purely algebraical treatment given below. It may be considered that even at this stage the continuity of the function should be investigated, or at least its differentiability. The rigorous proof, indeed, is not hard, depending merely on the inequality: When m and n are integers and m greater than n, $(a^m-1)/m$ is greater than $(a^n-1)/n$, and its generalisation. Instead of this, however, I suggest (if the

matter is thought essential) a semi-rigorous proof which is on a par with the proof of the existence of an area under a graph in a specialised case, as mentioned above.

(1) Since $$\frac{a^h-1}{h}=\frac{a^{\frac{1}{2}h}-1}{\frac{1}{2}h}\cdot\frac{a^{\frac{1}{2}h}+1}{2}>\frac{a^{\frac{1}{2}h}-1}{\frac{1}{2}h},$$
if h is positive; we have a descending sequence of positive members, and thus $(a^h-1)/h$ tends to a positive limit as h tends to zero *in this way.*

(2) It follows immediately that a^h tends to the limit 1, as h tends to zero in the same manner. Hence a^{x+h} tends to the limit a^x, its true value for $x=x$, both from above and below; hence a^x is continuous.

(3) The differential coefficient, the limit of $a^x\cdot\dfrac{a^h-1}{h}$, exists, and is also continuous, in the same manner.

Personally, however, I should consider that the introduction of any proof of differentiability at this stage not only non-essential, but even out of place; and should go on to assume that a tangent could be drawn to the graph of a^x, for any value of a, at any point; at the same time warning the beginner that later, on revision, this assumption would have to be justified. That is to say, I should assume that if (x, y), $(x+h, y+k)$ were any two solutions of the equation $y=a^x$, then k/h tended to a limit as h tended to zero.

Making this assumption, let (x_1, y_1), (x_2, y_2), (x_3, y_3), etc., be solutions of the equation $y=a^x$. Let x_1, x_2, x_3, etc., each receive the *same increment* h, and let the corresponding values of y_1, y_2, y_3, etc., be y_1+k_1, y_2+k_2, y_3+k_3, etc.

Then $$\frac{y_1+k_1}{y_2+k_2}=\frac{a^{x_1+h}}{a^{x_2+h}}=\frac{a^{x_1}}{a^{x_2}}=\frac{y_1}{y_2}=\frac{k_1}{k_2};$$
and $y_1/k_1=y_2/k_2=y_3/k_3=$etc., in a similar way.

Hence $$\frac{k_1/h}{y_1}=\frac{k_2/h}{y_2}=\frac{k_3/h}{y_3}=\text{etc.}$$

Assuming that k_1/h, k_2/h, k_3/h, etc., have limits, these limits are attained simultaneously on account of the common denominator h.

Hence, $dy/dx=c\,.\,y$, for any value of y, where c is a constant and equal to the gradient at the point (0, 1); (*i.e.* the limit of $(a^h-1)/h$, if this has been mentioned).

That is, $$\frac{d}{dx}(a^x)=c\,.\,a^x \quad\text{or}\quad \int_{-\infty}^{x} a^u\,du=a^x/c.$$

If we denote by the letter e that value of a for which the constant $c=1$, we have
$$\frac{d}{dx}(e^x)=e^x \quad\text{and}\quad \int_{-\infty}^{x} e^u\,du=e^x.$$

Again, if $y=a^x$, $x=\log_a y$; and dx/dy is the reciprocal of dy/dx, *i.e.* is equal to $1/cy$.

Therefore $$\frac{d}{dy}(\log_a y)=1/cy \quad\text{or}\quad \int_1^y du/u=c\,.\log_a y,$$
where c has the same meaning as before.

Hence, if $c=1$, we have
$$\frac{d}{du}(\log_e u)=\frac{1}{u} \quad\text{and}\quad \int_1^z du/u=\log_e z,$$
whatever u and z may be.

Hence e may be defined by the equation $\displaystyle\int_1^e du/u=1.$

Finally, $$c \cdot \log_a x = \int^x du/u = \log_e x\,;$$
and therefore $c = \log_e x/\log_a x = \log_e a$; and this is the limit of $(a^h - 1)/h$.

At a later stage, using the integrals in definite form as definitions of exponentials and logarithms, the deduction of the fundamental laws give good examples on change of variable. Thus,

$$\log_a m = \frac{1}{c}\int_1^m du/u = \frac{1}{c}\int_n^{mn} dv/v, \quad \text{where } v = nu\,;$$

hence $$\log_a mn = \frac{1}{c}\int_1^{mn} du/u = \frac{1}{c}\int_1^n du/u + \frac{1}{c}\int_n^{mn} du/u = \log_a n + \log_a m.$$

Again, $$\log_e(1+x) = \int_1^{1+x} du/u = \int_0^x dv/(1+v), \quad \text{where } u = 1+v\,;$$
from which the logarithmic series may be obtained.

Further, the deduction of the exponential series (and the consequent evaluation of e) forms an excellent exercise on integration by parts. For we have

$$e^{-x} - 1 = \int_0^{-x} e^u du = \left| ue^u - \frac{u^2}{\underline{|2}} e^u + \frac{u^3}{\underline{|3}} e^u - \ldots - (-1)^n \frac{u^n}{\underline{|n}} e_u \right|_0^{-x} + (-1)^n \int_0^{-x} \frac{u^n}{\underline{|n}} e^u du.$$

The latter integral tends to zero as n is indefinitely increased; hence we have

$$e^{-x}\left(1 + \frac{x}{\underline{|1}} + \frac{x^2}{\underline{|2}} + \ldots \text{to } \infty\right) = 1\,;$$

which gives $$e = 1 + 1 + 1/\underline{|2} + 1/\underline{|3} + \text{etc.}$$ J. M. C.

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359. When Newton discovered universal gravitation he began by the observation of isolated facts which suggested the law. This ascent from particular effects to general causes he entitled analysis. Once possessed of the principle he applied it to explain the remainder of the phenomena, and this was his synthesis. Hooke, his contemporary, employed the same words in the same way, except that he reversed them; and to this hour, though ignorant of the disagreement, some follow Hooke and some follow Newton. The terms have been adopted into the vocabulary of education, to distinguish the plan of commencing with rules and thence deducing their consequences, from the system of beginning with details and proceeding up to rules. A few years ago two individuals of some distinction got into an argument, which grew to an altercation, about the proper method of teaching Arithmetic. One was for Analysis, the other was for Synthesis. A third person, who read with a judgment unheated by disputation, at last pointed out to them that they agreed in everything except a name, or the controversy might possibly have been raging still.—W. Elwin, *Quarterly Review*, vol. 84, p. 325, 1849.

360. The house in London (off Leicester Square) in which Sir Isaac Newton lived has, within the last month, been repaired throughout, and within the last week, we are sorry to say, stuccoed all over on the outside, so that its old Queen Anne and Sir Isaac Newton character has been completely destroyed. A shilling subscription among the Fellows of the Royal Society might have saved it from this desecration—and, what is more, re-pointed the whole brick-work so as to retain the original appearance of the building. In this house Newton lived from 1710 till his death in 1727. A small observatory at the top was erected by him—and still remains, though badly disfigured.—*Athenaeum*, Nov. 10, 1849, p. 1135.

THE RELATIVE ABILITIES IN MATHEMATICS OF BOYS AND GIRLS.

BY F. SANDON, M.A.

IN the Memorandum from the Girls' Schools' Committee of the Mathematical Association, published on pages 13-15 of the *Mathematical Gazette* for January, 1926, we are told that the Committee deprecates the use of the term "inferiority" as applied to girls' mathematical work by the Report on the Differentiation of Curricula between the Sexes. Further, the Girls' School Committee says: "Where boys and girls are given equal opportunities, as in some mixed schools, the consensus of opinion is that very little difference in capacity between boys and girls is shown in work up to Matriculation standard...." It seems desirable that the question should be put on a sound basis, and with this in mind the writer has endeavoured to draw on material available to him for a statistical inquiry into the points involved. For the last two years he has been engaged in teaching in a mixed school where the boys and girls are taught mathematics together by the same teacher in the lowest two forms, and the forms are chosen on a basis of general standard in school work. The marks are given below. In each case the marks are those obtained by equating the quartiles (by a modification of a method due to Prof. Nunn and given by the writer in the *Forum of Education* in papers in 1924 and 1925) to 40 and 60, in homework, weekly tests, and terminal exams. The two marks given are those obtained at the ends of the Christmas and Midsummer terms. Thus boy A in Form II (there is no Form I in this school) obtained 38 (scaled) in homework, 40 in tests, and 53 in exam., total 131. The advantage of this method is that the mean and the standard deviation of the mark-frequency-distributions are approximately the same in the various cases. The writer was the teacher of all the subjects given—in Form II geometry was in the hands of the art master. The composition of the form was changed slightly in the middle of the year in the case of Form II, and practically not at all in the case of Form III: six of the boys and four of the girls went at the end of the first year with the teacher from Form II to Form III: in each case the A division is concerned. The writer thinks that he had no prejudices about the relative merits of boys and girls; his only prejudices were (1) that general impressions were worthless; (2) that the question could only be solved by statistical methods; (3) that a sex difference one way or the other would not be surprising in view of the other sex differences, more strange *a priori*, that statistical biometry has revealed.

The numbers involved are too small for any sound statistical method to be used, but some rough indications may be obtained.

If we take the average of the figures given at the end of this paper we have:

	Form II.		Form III.		
	Arithmetic.	Algebra.	Arithmetic.	Algebra.	Geometry.
Boys	154·9, 159·5	165·3, 178·3	160·8, 163·2	166·3, 162·2	170·2, 168·5
Girls	149·0, 138·1	136·1, 127·6	138·4, 138·8	124·6, 143·5	139·1, 120·9

The ordinary elementary statistical theory indicates that a difference in the averages in any one subject of more than some 15 or 20 is significant. We notice that in all cases, save one only, there is therefore a significant difference, so that the cumulative effect is to establish definitely in this case that these boys and girls are not equally good in the subjects considered. In the first case (Form II, Arithmetic, Christmas Term) this is much as expected, as nearly all the children had been selected for free places on an examination in

Arithmetic and in English, so that presumably on admission the boys and girls were of equal merit. They had been taught in mixed classes in various elementary schools of the district prior to admission.

It may, however, be urged that for some reason or other the boys chosen were rather more above the average boy than the girls were above the average girl. Let us try to eliminate to some extent this source of error by taking the girls in each case so that their arithmetical capacity won the same average mark as did that of the boys. We can do this approximately by taking only the first ten girls of Form II (*a, b, c, d, f, g, h, k, l, n*) and the first seven girls (*b, d, e, f, k, l, n*) of Form III. The averages are:

	Form II.		Form III.		
	Arithmetic.	Algebra.	Arithmetic.	Algebra.	Geometry.
Boys	154·9, 159·5	165·3, 178·3	160·8, 163·2	166·3, 162·3	170·2, 168·5
Girls	155·4, 136·5	141·7, 127·2	160·5, 154·9	135·7, 156·3	151·1, 122·6

Again it will be seen that throughout the second form there is a significant difference between the boys and the girls: in the third form the arithmetic and the algebra at the end of the time gives the boys a pull, but not so great, taken alone, to be significant; in view of all the results, however, the differences are probably symptomatic.

Let us take the case most unfavourable to the boys—that of algebra in the third form, where the girls make apparently great strides. It may be noted that in the second half-year here the class were taught factors by a lady teacher-student under the supervision of the writer.

Using the same method, choose the first six girls of Form III (*b, e, f, g, h*, and the average of *a* and *k*), as they get the same average algebra mark at the end of the year as do the boys. The results are:

	Subject.		
	Arithmetic.	Algebra.	Geometry.
Boys - -	160·8, 163·2	166·3, 162·2	170·2, 168·5
Girls - -	158·9, 149·2	147·6, 161·8	148·2, 136·0

It is difficult to say what this means; the arithmetic and the algebra differences are less than 20, and may not be significant: it is hardly likely that the girls should fall off in arithmetic and pull up in algebra by about as much. It serves simply to remind us that this method is devised to deal with very limited material, and that it is not to be regarded as statistically conclusive. but only as indicative of possible tendencies. We can, however, I think, feel fairly justified in assuming that even for these algebraically bright girls they are behind the boys in geometry (which is here inductive and early transitional stage B).

In view of the amazing difference in the case of geometry, it seems worth while trying to choose our girls to be equal to the boys in the average in geometry in the first term. This attempt fails, for only one girl has more than the boys' average, and the next girl at once pulls us below this average; the one girl left falls grievously below the boys' average in everything else. The result is, of course, valueless, as a single instance is utterly untrustworthy.

The writer is anxious to investigate further the rather unexpected results suggested above, and if any reader of the *Mathematical Gazette* can collaborate in supplying data suitable for a proper inquiry he would be glad to receive such material for analysis. In spite of the fact that nearly every school has some system of marks, yet he has been able to find very little useful to him:

the results are often not comparable between form and form, or even between the various divisions of any one form in any one school. It is desirable that there should be data comparable between school and school and at different epochs, and thus we need definite tests of mathematical ability and attainments to be set to everyone throughout. Even then several precautions must be taken, and it is submitted that the considerations now to be mentioned may invalidate the preceding results. On such limited material as we have dealt with it is not possible to allow for these.

(1) There must be full allowance made in any discussion of sex difference of possible discrimination in social class. In the school in question there are many brothers and sisters, and there is probably no difference in the social class of the boys and girls as a whole; in some cases at this school perhaps the boys come to the school because there is no alternative in the neighbourhood, whereas there are rival girls' schools; on the other hand, some girls come to the school whose brothers make longer journeys and go to more famous old schools in London.

(2) There must be full allowance made for age. The age question has proved particularly difficult at the school in consideration for two reasons:

(*a*) No age allowance is made in the free-place examination, with the result that the children just short of their twelfth birthday have a pull—due to almost a year's schooling and to a year's natural intellectual growth and development—over those just over their eleventh birthday, and an age distribution for entrants is found like:

Form II. Age on 31.7.25.

10·0 –	10·4 –	10·8 –	11·0 –	11·4 –	11·8 –
0	4	15	19	39	57

This means that some bright children are debarred from admission and their places filled by older and duller children.

(*b*) The average age of the girls in the forms in question is some three months in excess of that of the boys. This means that of boys and girls who are regarded as intellectually equal at the beginning of any school year the boy is probably younger and has a greater I.Q. and will develop more than the girl during the year. Part of the increase found in the boys' marks from the first term to the next may be due to this cause. The children were roughly between 10 and 12 when they started in Form II and between 12 and 14 when they finished in Form III.

(3) There must be equally stringent selection of the scholars from the general population. It is difficult to say how far this is true in any case, and it is difficult to distinguish from considerations such as (1) above. But in the area in question a definite number of free places are offered to boys and a definite number to girls, and if in any one social class, as is probably true, the parents are not equally keen on boys and girls having a five or more years' secondary course, it may result in a lower standard being necessary for admission in one case, and the selection, as the writer has reason to think, is the case for his school, not equally strong in the two cases—boys and girls.

Conclusion. From a very limited number of cases the result appears to emerge that:

(1) In a particular division of a mixed school form the boys are better at mathematics than the girls.

(2) That if girls are chosen equal to the boys in some mathematics subject they are not equal to the boys in other subjects.

(3) The writer is aware of the liability of the above discussion to be subject to statistical fallacies, due to work with material based on insufficient data, and would be glad if readers of the *Mathematical Gazette* would supply him with further material appropriate for statistical analysis in this matter. Data based on standardised tests relating to siblings would be particularly valuable.

MARKS IN FORM II, 1923-1924.

Children in Form all the Year.	Arithmetic.			Algebra.		
	Christmas.	Midsummer.	Increase.	Christmas.	Midsummer.	Increase.
Boy *A*	131	155	+24	168	173	+5
B	117	154	+37	154	168	+14
C	185	207	+22	174	209	+35
D	156	173	+17	162	184	+22
E	201	175	−26	133	192	+59
F	193	196	+3	177	196	+19
G	128	145	+17	165	153	−12
H	149	172	+23	180	179	−1
K	199	120	−79	176	195	+19
L	90	98	+8	164	134	−30
Girl *a*	161	138	−23	146	142	−4
b	188	207	+19	171	183	+12
c	134	134	0	75	106	+31
d	169	117	−52	149	139	−10
e	114	105	−9	114	117	+3
f	159	127	−32	137	108	−29
g	180	170	−10	136	109	−27
h	138	136	−2	153	115	−38
k	138	118	−20	152	117	−35
l	133	147	+14	173	137	−36
m	120	187	+67	102	142	+40
n	154	71	−83	125	116	−9

MARKS IN FORM III, 1924-1925.

Children in Form all the Year.	Arithmetic.			Algebra.			Geometry.		
	Christmas.	Summer.	Increase.	Christmas.	Summer.	Increase.	Christmas.	Summer.	Increase.
Boy *A*	124	153	+29	201	171	−30	188	163	−25
B	139	192	+53	164	161	−3	137	102	−35
C	209	224	+15	199	222	+23	215	216	+1
D	205	172	−33	153	172	+19	183	176	−7
E	149	117	−32	183	177	−6	177	156	−21
F	100	127	+27	110	129	+19	116	144	+28
G	173	192	+19	168	138	−30	140	184	+44
H	223	217	−6	223	227	+4	190	213	+23
K	200	224	+24	190	225	+35	185	202	+17
L	137	162	+25	145	106	−39	189	184	−5
M	185	180	−5	191	190	−1	167	140	−27
N	106	110	+4	142	123	−19	128	169	+41
P	180	130	+50	177	167	−10	205	185	−20
Q	133	135	+2	111	105	−6	166	139	−27
R	150	114	−36	138	121	−17	167	155	−12
Girl *a*	128	102	−26	86	135	+49	108	122	+14
b	222	183	−39	169	204	+35	146	173	+27
c	113	142	+29	106	102	−4	128	128	0
d	139	170	+31	90	132	+42	126	105	−21
e	153	140	−13	144	162	+18	158	136	−22
f	155	166	+11	158	165	+7	158	161	+3
g	113	141	+28	158	141	−17	156	125	−31
h	106	94	−12	103	129	+26	114	93	−21
k	137	143	+6	113	135	+22	145	132	−13
l	139	139	0	119	132	+13	180	95	−85
m	78	103	+25	102	121	+19	105	87	−18
n	178	143	−35	157	164	+7	145	94	−51

FRANK SANDON.

QUESTIONNAIRE ON THE TEACHING OF MATHEMATICS IN EVENING CONTINUATION SCHOOLS.

From 16th Jan. to 20th March there was given, at University College, Nottingham, a course of ten two-hour classes in Mathematics and Drawing for Continuation School Teachers. This was attended by forty-eight teachers from various parts of the East Midlands. None of these was a member of the Mathematical Association, but the opening address gave an account of the Association's Report on the Teaching of Mathematics to Evening Technical Students. The first half-hour of each class was generally occupied by an address (by a different speaker each week) on some of the wider aspects of Mathematics. Then followed an hour's lecture by Mr. L. Turner, of Coventry Municipal Technical Institute. The last half-hour was usually devoted to a discussion.

At the end of the course a questionnaire was sent round. Most of the replies require no comment. The unanimous opinion in favour of the use of apparatus showed how much the teachers appreciated the great variety of ingenious home-made apparatus which was a special feature of Mr. Turner's lectures. It is interesting to notice that the introduction of Deductive Geometry was approved with only one dissentient. In fact, the recommendations of the Mathematical Association's Report seemed to be generally approved, with one important apparent exception. By 26 to 14, it was voted that practical applications should be taught at the same time as algebraic processes, rather than that the latter should be thoroughly mastered first. The wording of this question was perhaps unfortunate. The choice was between two alternatives, neither of which was what the Report recommended. I think that I am correct in stating that the members of the sub-committee that drew up the Report never contemplated that *all* practical applications should be postponed until *all* algebraic processes had been mastered, but merely wished that each separate process (*e.g.* the manipulation of indices) should at first be practised, in simple cases, until it could be performed with reasonable ease and accuracy, and then applied to practical problems. After this had been done with one process the same sequence could be gone through with another.

As a result of this valuable course of lectures, the Report has been brought to the notice of an important class of teachers quite out of touch with our Association. Our best thanks are due to Mr. Turner for this work, and also for allowing the publication of the replies to the questionnaire. H. T. H. P.

QUESTIONNAIRE.

1. Have you found that a "vocational bias" to Mathematics tends to increase the interest of the student?	Yes	40
	No	0
2. Do you find that the majority of your students attend Mathematics classes solely that they may gain an advantage (promotion at their work)?	Yes	34
	No	3
	Non-Committal	3
3. Have you had difficulty in finding a suitable Mathematics Text Book?	Yes	31
	No	9
4. Do you find that "individual" rather than "class" teaching meets the requirements of evening school students?	Yes	32
	No	6
	Non-Committal	2
5. Does your present Mathematics scheme include the teaching of Deductive Geometry?	Yes	17
	No	23
6. Are you agreed that it is desirable to include a course of deductive geometry in Evening School Mathematics?	Yes	39
	No	1

7. Do you think the adoption of the Metric System in this country is desirable ?	Yes	40
	No	0
8. Would a standardised system of teaching fundamental operations such as subtraction and multiplication be an advantage ?	Yes	31
	No	9
9. Do you think that the use of apparatus is a help in the teaching of Mathematics ?	Yes	40
	No	0
10. Do you think that practical applications should be taught at the same time as say algebraic processes, or that the latter should be thoroughly mastered first ?	Former	26
	Latter	14
11. Are you in favour of the "Examination System" in Evening Continuation Schools ?	Yes	20
	No	18
	Non-Committal	2

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361. For his sake she watches the post, gets to bed and contemplates, skims the news for tidings of his ship the *Dunkirk*, and sleeps ill when the wind is high. Finally she even goes so far as to begin the study of Euclid.

"Attacked Euclid," several entries in the journal run, "drawed some of the figures, a little dull about an angle not to say a good deal so. Think I shall like the kind of thing," she adds hopefully, "and much more for a certain reason. Heigh! Ho! Nothing of the *Dunkirk*."...

Read news that Captain Howe was left in command of a squadron in the Bay of Biscay. On January 14, she "worked and attacked Euclid," and was "immensely dull." On the 15th she was "eat up with vapours." On the 20th "she did nothing and heard nothing," and was again "immensely dull," and on the 21st she was "more out of spirits than ever." On the 22nd she "tried Euclid again in rather better spirits"; but on the 29th the blow falls. No more Euclid for Elizabeth Raper! Captain Howe is not in the Bay of Biscay. He has been "in Town" the whole time.—*The Receipt Book of Elizabeth Raper.*

362. The late Sir Laurence Gomme used to recall how he gained a tip in the application of music to work from the old Lancashire weavers, who invariably crooned at their looms; and (he) confessed that for many years it was his habit to cast up columns of figures to the tune of Gregorian chants, which he found an aid to rapidity and accuracy.—*Manchester Guardian*, Dec. 5. 1924 [per Mr. H, J. Woodall.]

363. The mere fact that music and mathematics should be allied is a kind of phenomenon. One can hardly believe how Euclid and Jenny Lind should have any common bond of union; but deep in the secret caverns of the mind the materials from which both are supplied mingle in one common source, and the paths which have conducted a Galileo, a Kepler, and a Herschel to the profoundest abstractions the human mind is capable of, have started from the sweet portals of musical sound.—*Quarterly Review*, vol. 83, p. 484.

364. John Dominic Cassini was cured of his belief in astrology by the success of a prediction based upon calculations which proved to be incorrect. He speaks of himself as the first who observed the variation of the moon's diameter depending upon her altitude. "Oui," says Delambre, in one of his parentheses, "le premier après Kepler, Auzout et Hevelius."

365. The importance (of the problem of the duplication of the cube) declined with the rise of the decimal arithmetic. Many different attempts were made ... Any process for the solution was called *mesolabum* (a term as old as Vitruvius). One of the last was that of the celebrated Vieta, containing an error, which is the more remarkable, that little, if any, notice has ever been taken of it. (See his *Works*, Schooten's edn., p. 273.)—De Morgan, "Duplication of the Cube," *Penny Cyclopaedia.*

MATHEMATICAL NOTES.

830. [U. 10.] *A Simplification.*

On p. 84 of Prof. Eddington's *Mathematical Theory of Relativity*, the following somewhat discouraging remark occurs—" omitting the terms (223 in number) which now obviously vanish." It is proposed in this article to tie the terms up into parcels, as it were, so that their vanishing is made even more obvious.

Since all the g's vanish except those which have like suffixes, the equations of the geodesic become

$$\frac{d^2x_\sigma}{ds^2}=\frac{1}{2g_{\sigma\sigma}}\left(\frac{\partial g_{\mu\sigma}}{\partial x_\nu}+\frac{\partial g_{\sigma\nu}}{\partial x_\mu}-\frac{\partial g_{\mu\nu}}{\partial x_\sigma}\right)\frac{dx_\mu}{ds}\frac{dx_\nu}{ds}\equiv(\mu\nu,\sigma)\frac{dx_\mu}{ds}\frac{dx_\nu}{ds},$$

so that $(\mu\nu, \alpha)$ replaces $\{\mu\nu, \alpha\}$ in $G_{\mu\nu}$.

When μ, ν, α represent different suffixes,

$$(\mu\mu,\mu)=\frac{1}{2}\frac{\partial l_\mu}{\partial x_\mu};\ (\mu\alpha,\alpha)=\frac{1}{2}\frac{\partial l_\alpha}{\partial x_\mu};\ (\mu\mu,\alpha)=-\frac{1}{2g_{\alpha\alpha}}\frac{\partial g_{\mu\mu}}{\partial x_\alpha};\ (\mu\nu,\alpha)=0,$$

where $l_\mu\equiv\log g_{\mu\mu}$.

Also $\sum\limits_{\alpha=1}^{\alpha=4}(\mu\alpha,\alpha)\equiv\frac{1}{2}\frac{\partial l}{\partial x_\mu}$, when $l=\sum\limits_1^4 l_\alpha$.

It will be convenient to refer to the four suffixes as μ, ν, ρ, σ and to use α and β as dummy suffixes.

$$G_{\mu\nu}=\frac{1}{2}\frac{\partial^2 l}{\partial x_\mu\partial x_\nu}-\frac{\partial}{\partial x_\beta}(\mu\nu,\beta)-(\mu\nu,\alpha)\frac{1}{2}\frac{\partial l}{\partial x_\alpha}+(\mu\beta,\alpha)(\nu\alpha,\beta).$$

(*a*) When $\nu\neq\mu$,

$$\begin{aligned}G_{\mu\nu}=&\frac{1}{2}\frac{\partial^2 l}{\partial x_\mu\partial x_\nu}-\frac{\partial}{\partial x_\mu}(\mu\nu,\mu)-(\mu\nu,\mu)\frac{1}{2}\frac{\partial l}{\partial x_\mu}+(\mu\nu,\mu)(\nu\mu,\nu)\\&-\frac{\partial}{\partial x_\nu}(\mu\nu,\nu)-(\mu\nu,\nu)\frac{1}{2}\frac{\partial l}{\partial x_\nu}+(\mu\mu,\nu)(\nu\nu,\mu)+(\mu\alpha,\alpha)(\nu\alpha,\alpha)\\=&\frac{1}{2}\frac{\partial^2 l}{\partial x_\mu\partial x_\nu}-\frac{1}{2}\frac{\partial^2(l_\mu+l_\nu)}{\partial x_\mu\partial x_\nu}-\frac{1}{4}\frac{\partial l_\mu}{\partial x_\nu}\frac{\partial l}{\partial x_\mu}-\frac{1}{4}\frac{\partial l_\nu}{\partial x_\mu}\frac{\partial l}{\partial x_\nu}\\&+\frac{1}{2}\frac{\partial l_\mu}{\partial x_\nu}\cdot\frac{\partial l_\nu}{\partial x_\mu}+\frac{1}{4}\frac{\partial l_\mu}{\partial x_\mu}\cdot\frac{\partial l_\mu}{\partial x_\nu}+\frac{1}{4}\frac{\partial l_\nu}{\partial x_\mu}\cdot\frac{\partial l_\nu}{\partial x_\nu}+\frac{1}{4}\frac{\partial l_\rho}{\partial x_\mu}\cdot\frac{\partial l_\rho}{\partial x_\nu}+\frac{1}{4}\frac{\partial l_\sigma}{\partial x_\mu}\cdot\frac{\partial l_\sigma}{\partial x_\nu}\\=&\left(\frac{1}{2}\frac{\partial^2}{\partial x_\mu\partial x_\nu}-\frac{1}{4}\frac{\partial l_\mu}{\partial x_\nu}\cdot\frac{\partial}{\partial x_\mu}-\frac{1}{4}\frac{\partial l_\nu}{\partial x_\mu}\cdot\frac{\partial}{\partial x_\nu}+\frac{1}{4}\frac{\partial}{\partial x_\mu}\cdot\frac{\partial}{\partial x_\nu}\right)(l_\rho+l_\sigma).\end{aligned}$$

(*b*) When $\nu=\mu$,

$$G_{\mu\mu}=\frac{1}{2}\frac{\partial^2 l}{\partial x_\mu^2}-\frac{\partial}{\partial x_\beta}(\mu\mu,\beta)-\frac{1}{2}(\mu\mu,\alpha)\frac{\partial l}{\partial x_\alpha}+(\mu\beta,\alpha)(\mu\alpha,\beta).$$

Collect the terms in μ only and those in α only, where now α is summed for values not equal to μ. Then

$$\begin{aligned}G_{\mu\mu}=&\frac{1}{2}\frac{\partial^2 l}{\partial x_\mu^2}-\frac{\partial}{\partial x_\mu}(\mu\mu,\mu)-\frac{1}{2}(\mu\mu,\mu)\frac{\partial l}{\partial x_\mu}+(\mu\mu,\mu)^2+(\mu\alpha,\alpha)^2\\&-\frac{\partial}{\partial x_\alpha}(\mu\mu,\alpha)-\frac{1}{2}(\mu\mu,\alpha)\frac{\partial l}{\partial x_\alpha}+2(\mu\mu,\alpha)(\mu\alpha,\mu)\\=&\frac{1}{2}\frac{\partial^2 l}{\partial x_\mu^2}-\frac{1}{2}\frac{\partial^2 l_\mu}{\partial x_\mu^2}-\frac{1}{4}\frac{\partial l_\mu}{\partial x_\mu}\cdot\frac{\partial l}{\partial x_\mu}+\frac{1}{4}\left(\frac{\partial l_\mu}{\partial x_\mu}\right)^2+\frac{1}{4}\left(\frac{\partial l_\alpha}{\partial x_\mu}\right)^2\\&+\frac{\partial}{\partial x_\alpha}\left(\frac{1}{2g_{\alpha\alpha}}\frac{\partial g_{\mu\mu}}{\partial x_\alpha}\right)+\frac{1}{4g_{\alpha\alpha}}\frac{\partial g_{\mu\mu}}{\partial x_\alpha}\frac{\partial l}{\partial x_\alpha}-\frac{1}{2g_{\alpha\alpha}}\frac{\partial g_{\mu\mu}}{\partial x_\alpha}\cdot\frac{\partial l_\mu}{\partial x_\alpha}.\end{aligned}$$

In this, the first line is equal to

$$\sum_{a\neq\mu}\left[\frac{1}{2}\frac{\partial^2 l_a}{\partial x_\mu^2}-\frac{1}{4}\frac{\partial l_\mu}{\partial x_\mu}\frac{\partial l_a}{\partial x_\mu}+\frac{1}{4}\left(\frac{\partial l_a}{\partial x_\mu}\right)^2\right]=\sqrt{\frac{g_{\mu\mu}}{g_{aa}}}\frac{\partial}{\partial x_\mu}\left(\frac{1}{\sqrt{g_{\mu\mu}}}\frac{\partial\sqrt{g_{aa}}}{\partial x_\mu}\right).$$

In the second line, taking the first two terms together, and putting $g\equiv g_{11}g_{22}g_{33}g_{44}$, we obtain

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x_a}\left(\frac{\sqrt{g}}{2g_{aa}}\frac{\partial g_{\mu\mu}}{\partial x_a}\right)-\left(\frac{\sqrt{g}}{2g_{aa}}\cdot\frac{\partial g_{\mu\mu}}{\partial x_a}\right)\frac{\partial l_\mu}{\partial x_a}\cdot\frac{1}{\sqrt{g}},$$

i.e.

$$\frac{g_{\mu\mu}}{\sqrt{g}}\frac{\partial}{\partial x_a}\left(\frac{\sqrt{g}}{g_{aa}}\frac{\partial\log\sqrt{g_{\mu\mu}}}{\partial x_a}\right).$$

Thus altogether,

$$G_{\mu\mu}=\sqrt{\frac{g_{\mu\mu}}{g_{aa}}}\frac{\partial}{\partial x_\mu}\left(\frac{1}{\sqrt{g_{\mu\mu}}}\frac{\partial\sqrt{g_{aa}}}{\partial x_\mu}\right)+\frac{g_{\mu\mu}}{\sqrt{g}}\frac{\partial}{\partial x_a}\left(\frac{\sqrt{g}}{g_{aa}}\cdot\frac{\partial\log\sqrt{g_{\mu\mu}}}{\partial x_a}\right).$$

We require the case when g_{11}, g_{22} and g_{44} are functions of x_1 only, and g_{33} a function of x_1 and x_2.

Thus
$$G_{11}=\sum_2^4\sqrt{\frac{g_{11}}{g_{aa}}}\frac{\partial}{\partial x_1}\left(\frac{1}{\sqrt{g_{11}}}\frac{\partial\sqrt{g_{aa}}}{\partial x_1}\right),$$

$$G_{22}=\sqrt{\frac{g_{22}}{g_{33}}}\frac{\partial}{\partial x_2}\left(\frac{1}{\sqrt{g_{22}}}\frac{\partial\sqrt{g_{33}}}{\partial x_2}\right)+\frac{g_{22}}{\sqrt{g}}\frac{\partial}{\partial x_1}\left(\frac{\sqrt{g}}{g_{11}}\frac{\partial\log\sqrt{g_{22}}}{\partial x_1}\right),$$

$$G_{33}=\frac{g_{33}}{\sqrt{g}}\left[\frac{\partial}{\partial x_1}\left(\frac{\sqrt{g}}{g_{11}}\frac{\partial\log\sqrt{g_{33}}}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(\frac{\sqrt{g}}{g_{22}}\cdot\frac{\partial\log\sqrt{g_{33}}}{\partial x_2}\right)\right],$$

$$G_{44}=\frac{g_{44}}{\sqrt{g}}\frac{\partial}{\partial x_1}\left(\frac{\sqrt{g}}{g_{11}}\cdot\frac{\partial\log\sqrt{g_{44}}}{\partial x_1}\right);$$

while from (*a*), $G_{\mu\nu}\equiv 0$, when $\mu\neq\nu$, except

$$G_{12}=\frac{1}{2}\frac{\partial^2 l_3}{\partial x_1\partial x_2}-\frac{1}{4}\frac{\partial l_1}{\partial x_2}\frac{\partial l_3}{\partial x_1}-\frac{1}{4}\frac{\partial l_2}{\partial x_1}\frac{\partial l_3}{\partial x_2}+\frac{1}{4}\frac{\partial l_3}{\partial x_1}\cdot\frac{\partial l_3}{\partial x_2}.$$

Making the substitutions $g_{11}=-e^\lambda$, $g_{22}=-x_1^2$, $g_{33}=-x_1^2\sin^2 x_2$, $g_{44}=e^\nu$, where λ and ν are functions of x_1, Eddington's equations 38·61 to 38·65 follow at once.

N. M. GIBBINS.

831. [K¹. 2. c.] *A Property of the Nine-Points Circle.*

Three points and the orthocentre of the corresponding triangle form a set of 4 points $A_1, \ldots A_4$ (sometimes called "orthocentric"), such that each is the orthocentre of the triangle formed by the other three. The nine points, from which the N.P.C. of any one of the triangles derives its name, can be defined symmetrically as the mid-points of the 6 sides $A_1A_2, \ldots$ and the intersections of the three pairs of opposite sides, such as A_1A_2, A_3A_4. It follows at once that the four triangles $A_1A_2A_3, \ldots$, have a common N.P.C., which is touched by 16 circles, the inscribed and escribed circles of each of the four triangles.

All this is well known, but, though it would be very rash to claim novelty for any result in so well-worn a subject, I have not seen anywhere the results which follow.

If $S_1, \ldots, S_4$ are the circumcentres of the triangles $A_2A_3A_4, \ldots$, then reciprocally $A_1, \ldots$ are the circumcentres of the triangles $S_2S_3S_4, \ldots$; the points S form an orthocentric set; the two quadrangles $A_1 \ldots A_4$ and $S_1 \ldots S_4$ are congruent; the 8 triangles formed by any three points A or by any three points S have a common N.P.C., which is accordingly touched by 32 circles, the inscribed and escribed circles of the eight triangles.

The proof follows immediately from the familiar property that the centre N of the N.P.C. is the mid-point of the line joining the orthocentre and circum-

centre of a triangle, so that a rotation about N through 180° interchanges the points A with the points S, without altering the N.P.C.

It might interest some of your readers with a taste for geometrical drawing to make a diagram showing the N.P.C. and the 32 tangent circles.

King's College, Cambridge. ARTHUR BERRY.

832. [L¹. 1. a.] *The Reduction of the Equation of a Central Conic to its Simplest Form.*

In Note No. 813 Mr. Dobbs gives a method of reducing the conic

$$ax^2+2hxy+by^2=1$$

to the form $$(a+mh)X^2+(b-mh)Y^2=1,$$

where $m^2h+m(a-b)-h=0$, which form Mr. Dobbs says is suggested by the argument in Note No. 796.

He shows how to reduce the coefficient of X^2 from the form

$$(a+2mh+m^2b)/(1+m^2)$$

to the form $(a+mh)$ by means of the equation for m.

If this form, which is not a well-known one, has arisen out of another investigation, it can hardly be the result of an obvious line of procedure. Is it not more likely that a student would first solve the equation for m and then leave the coefficients of X^2 and Y^2 in the forms

$$\tfrac{1}{2}\{a+b\pm\sqrt{(4h^2+(a-b)^2)}\}\ ?$$

Mr. Dobbs points out a short road to the reduction. His method is a very good one, and quite free from ambiguity.

In Note No. 814 Mr. Child gives the well-known procedure depending on the consideration of a circle concentric to the conic. This amounts to introducing into the argument extraneous matter which is not suggested by the problem to be solved.

It is to be noted in the first place that this circle is sometimes real and sometimes imaginary. The use of the imaginary circle in the case of the hyperbola must constitute a difficulty to the thoughtful student.

Further, it is only because it is known *otherwise* that if the radii of the two circles which appear are, say, r_1, r_2, whether real or imaginary, then the equation of the conic must be $x^2/r_1^2+y^2/r_1^2=1$, *there being no xy term on the left-hand side of the equation*, that the method is useful. It is a method of obtaining the results of the transformation without actually performing it. *It is not a demonstration of the transformation.*

The position of the axes is determined without ambiguity by this method.

Mr. Child in conclusion says that the case in which both values of r^2 are negative cannot arise.

He proves the equation

$$(Ax+Hy)^2+(Hx+By)^2+(AB-H^2)(x^2+y^2)=A+B.$$

Then he says that if both values of r^2 are negative, both $(A+B)$ and (H^2-AB) are negative, and the equation last written cannot be satisfied.

But the equation only means that it cannot be satisfied by any *real* values of x and y.

So the conclusion is that if the conic be *real*, then both values of r^2 cannot be negative.

It may be noted that if the equation of the general conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

is written down at random, and if it should happen that all the coefficients are real and such that $(ab-h^2)$ and $a(abc-af^2-bg^2-ch^2+2fgh)$ are both positive, then the conic is an imaginary one, and both values of r^2 are negative.

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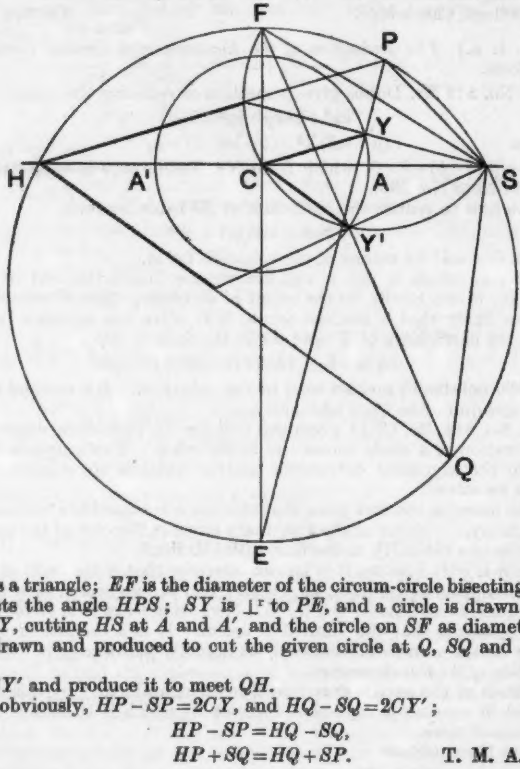
833. [K^1. 8. c.] *Suggested by:—HP, PS are chords of a circle. Required to construct in the circle a quadrilateral such that HP, PS, being two of the sides, a circle can be inscribed in it.*



HPS is a triangle; EF is the diameter of the circum-circle bisecting HP at C; EP bisects the angle HPS; SY is $\perp^r$ to PE, and a circle is drawn centre C, radius CY, cutting HS at A and A', and the circle on SF as diameter at Y'; FY' is drawn and produced to cut the given circle at Q, SQ and QH being joined.

Join SY' and produce it to meet QH.

Then, obviously, $HP-SP=2CY$, and $HQ-SQ=2CY'$;

hence $$HP-SP=HQ-SQ,$$

or $$HP+SQ=HQ+SP.$$

T. M. A. COOPER.

834. [R. 6. a. β; 9. b.] *Note on Note* 698, *Gazette* xii. p. 61.

By using Gibbs and Heaviside's idea of a scalar product of two vectors and the laws of such scalar products, the note in question can be obtained in a few steps.

I. $$(m+m')(mU^2+m'U'^2) \equiv (mU+m'U')^2+mm'(U-U')^2.$$

The above identity is easily verified when U and U' are algebraical numbers. Here, of course, U and U' are vectors, and the squares are scalar products of a vector by itself.

But $mU+m'U'=MV$, where, of course, the summation is not algebraical but vectorial.

$$\therefore\ (m+m')\{mU^2+m'U'^2\} \equiv M^2V^2+mm'(U-U')^2;$$

$$\therefore\ \tfrac{1}{2}(mU^2+m'U'^2-MV^2)=\frac{mm'}{2(m+m')}(U-U')^2,$$

which is the result of the note.

It may be noted that the above identity I. is very convenient for proving the loss of K.E. mentioned in the latter part of the note.

From $(a^2+b^2)(p^2+q^2)=(ap+bq)^2+(aq-bp)^2$

and $(a^2+b^2+c^2)(p^2+q^2+r^2)=(ap+bq+cr)^2+(aq-bp)+(br-cq)^2+(ar-cp)^2$,

we easily deduce

II. $(m_1+m_2)(m_1u_1^2+m_2u_2^2)=(m_1u_1+m_2u_2)^2+m_1m_2(u_1-u_2)^2$.

III. $(m_1+m_2+m_3)(m_1u_1^2+m_2u_2^2+m_3u_3^2)$

$$=(m_1u_1+m_2u_2+m_3u_3)^2+m_1m_2(u_1-u_2)^2+m_2m_3(u_2-u_3)^2 +m_1m_3(u_1-u_3)^2.$$

In the above identities, the summations may be considered vectorial or algebraical, and the products scalar or algebraical.

Let $v_1, v_2, \ldots$, be velocity vectors after impact.

By Newton's law

component of v_1-v_2 along line of centres

$=(-e)\times$ component u_1-w_2 along line of centres.

Also for smooth bodies, components of velocities perpendicular to line of centres are unchanged by impact.

It can then be easily proved that

$(v_1-v_2)^2-(u_1-u_2)^2=(1-e^2)\times$(component of u_1-u_2 along line of centres)^2.

Similar to identity II., we have

IV. $(m_1+m_2)(m_1v_1^2+m_2v_2^2)=(m_1v_1+m_2v_2)^2+m_1m_2(v_1-v_2)^2$.

But $m_1v_1+m_2v_2=m_1u_1+m_2u_2$ (vectorial addition).

$\therefore$ subtracting II. from IV.,

$$(m_1+m_2)\{m_1v_1^2+m_2v_2^2-m_1u_1^2-m_2u_2^2\}$$
$$=m_1m_2\{(v_1-v_2)^2+(u_1-u_2^2)\}$$
$$=m_1m_2\{\text{component of } (u_1-u_2) \text{ along line of centres}\}^2\times(1-e^2);$$

$\therefore$ $\frac{1}{2}(m_1v_1^2+m_2v_2^2-m_1u_1^2-m_2u_2^2)$

$$=(1-e^2)\,\frac{m_1m_2}{2M}\text{ (component of } u_1-u_2 \text{ along line of centres).}$$

Here v_1, v_2, etc., are vectors.

By using identity III., we can get similar results for impact of three bodies or bursting of shell into three pieces. The result can be extended to impact of four bodies by using Euler's well-known expansion of

$$(a^2+b^2+c^2+d^2)(p^2+q^2+r^2+s^2)$$

as sum of squares, which is most easily proved by taking product of two determinants of second order. S. PURUSHTHAM.

Vizianagram, India.

835. [V. 1. a. λ.] *Note on Mr. Bickley's article* (*Gazette* 180, *page* 10).

It is shown that the solution of $\frac{dy}{dx}=ky$ is

$$Y=Y_0\left(1+kx+\frac{k^2x^2}{2\,!}+\frac{k^3x^3}{3\,!}+\ldots\right), \quad \ldots\ldots(1)$$

where Y_0 is a constant, which we may take, for our purpose, as unity.

Now suppose $$\frac{dy}{dx}=Y. \quad \ldots\ldots(2)$$

Then $$\frac{dy^k}{dx}=ky^{k-1}\cdot\frac{dy}{dx}$$
$$=ky^k, \quad by\ (2).$$

This is the equation we solved in (1); hence

$$y^k = 1 + kx + \frac{k^2x^2}{2\,!} + \frac{k^3x^3}{3\,!} + \dots, \quad \dots\dots\dots\dots\dots\dots\dots(3)$$

or

$$\left(1 + x + \frac{x^2}{2\,!} + \frac{x^3}{3\,!} + \dots\right)^k = 1 + kx + \frac{k^2x^2}{2\,!} + \dots .$$

Let $x = 1$, and the series on the left of the equation becomes 2·71828... or e.

Then $$e^k = 1 + k + \frac{k^2}{2\,!} + \frac{k^3}{3\,!} + \dots .$$

F. W. HARVEY.

836. [X. 4.] P. 505. I have been asked for a key to the solution given. The simplest method seems to be:

Draw BQ parallel to Op_1p_2 to cut AC, produced, in Q; then PQ along the road would take the same time as PB across the field, whether P is P_1 or P_2: from the similar triangles OpB, QBP.

A. LODGE.

837. [B. 1. b.] Note on Note 810, *Gazette*, xii. 506. *The Sign of a Term in the Expansion of a Determinant.*

The following rules for finding the sign of any term in the expansion of a determinant can be applied very quickly.

(a) Write down the numbers of the rows in which the factors occur in the several columns. *E.g.* 5 7 1 2 3 6 4.

(b) Underline one of the numbers, *e.g.* 1. Note the number occupying the first place, here 5. Put a dot under the 5 and note the number occupying the 5th place, here 3. Put a dot under the 3. In this case the number in the third place is already underlined, and we now have the distribution:

$$\underset{\cdot}{5} \quad 7 \quad \underline{1} \quad 2 \quad \underset{\cdot}{3} \quad 6 \quad 4.$$

(c) Proceeding in the same way, we underline the 2 and dot 7 and 4. Finally we underline the 6, which is in the sixth place, and get

$$\underset{\cdot}{5} \quad \underset{\cdot}{7} \quad \underline{1} \quad \underline{2} \quad \underset{\cdot}{3} \quad \underline{6} \quad \underset{\cdot}{4}.$$

(d) The sign of the term is + or − as the number of dots is even or odd. In the example it is +.

Classification of Mathematical Puzzles. The Chinese 15-puzzle.

In the puzzle, which is sold as the Chinese 15-puzzle, there are 15 numbered blocks or "stones" in a square box with room for 16. "The stones are to be placed unarranged in the little box and shoved in proper order." It is an even chance whether the puzzle can be solved or not. To find whether a particular arrangement will give a solution we may apply the foregoing rules (disregarding rows and columns and thinking merely of the position each block should occupy). Square number 16 being blank to begin with, the criterion for a solution is an even number of dots or an odd number of bars. Thus of the initial arrangements:

$$\begin{array}{cccc} \underset{\cdot}{7} & \underset{\cdot}{5} & \underset{\cdot}{13} & \underline{1} \\ \underset{\cdot}{11} & \underset{\cdot}{8} & \underset{\cdot}{3} & \underset{\cdot}{9} \\ \underline{6} & \underset{\cdot}{4} & \underline{2} & \underset{\cdot}{15} \\ \underset{\cdot}{14} & \underset{\cdot}{10} & \underline{12} & \end{array} \qquad\qquad \begin{array}{cccc} \underset{\cdot}{7} & \underset{\cdot}{5} & \underset{\cdot}{13} & \underset{\cdot}{14} \\ \underset{\cdot}{11} & \underset{\cdot}{8} & \underset{\cdot}{3} & \underset{\cdot}{9} \\ \underline{6} & \underline{4} & \underline{2} & \underset{\cdot}{15} \\ \underline{1} & \underset{\cdot}{10} & \underline{12} & \end{array}$$

The latter allows of a solution of the puzzle, the former does not.

F. J. W. W.

REVIEWS.

Klassische Stücke der Mathematik. By ANDREAS SPEISER. Pp. 168. 8 Swiss frs.; bd. 12 frs. 1925. (Orell Füssli, Zürich and Leipzig.)

"Wonderful, it seems to me, the insight which the mathematicians have gained!" Archytas of Tarentum.

Such is the old-world motto which Professor Speiser of Bâle puts on the title-page of his collection of chosen passages translated into German from the writings of great thinkers of all time and of many European lands.

The following is the table of Contents:—1. Harmony, Archytas. 2. Description of the Earth. Plato. 3. Mathematics and Metaphysics. Plato. 4. The essence of mathematics. Plato. 5. The Aristotelian idea of space. 6. Euclidean Space. 7. Integration. Archimedes. 8. The end of space and the Beyond. Dante. 9. Proportion in Art. Leonardo da Vinci. 10. The Paradise of Tintoretto (Frontispiece). 11. The influence of the stars. Kepler. 12. Goethe on Descartes. 13. Analytical Geometry. Descartes. 14. Mathematical thinking. Pascal. 15. The law of the large number. Jakob Bernoulli. 16. The postulate of parallels. Hieronymus Saccheri. 17. From the Social Contract of J. J. Rousseau. 18. Theory of numbers. Euler. 19. The bridges of Königsberg. Euler. 20. The Kinetic theory of gases. Daniel Bernoulli. 21. The theory of relativity. Einstein. 22. Modern views on mathematics. Sylvester. 23. Natural geometry. J. Hjelmslev. 24. Plato on the learned.

Moulded on the pattern of those musical albums put into the hands of the amateur, this little book has at once the charm and the weakness of an anthology. It is not what is called a mathematical book in the ordinary sense; on the other hand, it is only to one who loves and is versed in mathematics that it could appeal. We could wish to see the example set by Professor Speiser followed in our own country. It is indeed well that the mathematician should have before him in his own tongue the actual words used by great philosophers, poets and artists, expressing their views as to the nature and aims of mathematics in the various domains of our mental life.

But Professor Speiser appeals to a wider audience. "Every person of cultivation now-a-days," he says, "has been taught mathematics in his youth for some twelve years, perhaps three times a week. . . . Yet in wide circles of the cultured world we meet complete misconception of the essence of mathematics, and not infrequently a certain repugnance to mathematical science." And he adds, though we cannot agree with him:—"The cause cannot be found in the teaching itself, for at the present day this is quite first-rate.." In Professor Speiser's opinion the fault lies in the failure to bring out the connexion between mathematics and the other branches of knowledge, and he sums up in the words of Leibniz "Mathematics is the science of the imaginable."

Einleitung in die Mengenlehre, eine elementare Einführung in das Reich des Unendlichgrossen. By A. FRAENKEL. Pp. 225. 2·60$. 1923. (Springer, Berlin.)

The first edition of this book, bearing the title "An Introduction to the Theory of Sets; a guide to the realm of Infinite Magnitude, comprehensible to everyone," appeared in 1919, with a striking preface, in which the writer sketched days passed by him in the trenches, wiling away weary hours by chat about one of the deepest and newest branches of mathematics to the listening Field-Greys around him. It was a handy volume of 156 pages, thick paper with wide margins and good print, in a cover of cream-coloured sugar-paper. This tempting exterior corresponded to the interior. Here was light reading for the trained mathematician, even for one who, in the course of a busy life, had forgotten the details of the science; and, for the non-mathematical thinker—conversant perchance with Aristotle's dictum "Infinite magnitudes do not exist"—the first chapters, at any rate, written as was promised in easy language, seemed to afford a glimpse into the forbidden forest of thought.

It is characteristic of the Germany of to-day in her best aspect that in three years the book was out of print. What we have in our hands is the second edition, masquerading as a text-book in the ugly orange cover of Professor Courant's series on the Foundations of Mathematical Science.

The book has been considerably enlarged (pp. 251), and revised in detail.* The new chapters, entitled "Objections to the Theory of Sets" and "The axiomatic building-up of the Theory of Sets," will be useful, all the more that this is the department in which the author actually works. The very tone in which they are written also will warn the reader against taking what they contain as in any sense final, and will, it may be hoped, stimulate the student to consult the authors quoted, although apparently not always digested, by Fraenkel himself. In the new preface the author modestly deprecates criticism on this score, more especially with regard to non-German writing, and we cannot do more than express the hope that this talented writer will, before the book gets into a third edition, undertake a study of at least the English works, some of which are already classical, on the Theory of Sets. He will then assuredly be induced to rewrite the chapter entitled "Linear Sets of Points," which is the chief flaw in an artistic production. A footnote, twice appended, says that this chapter may be omitted without affecting the later part of the book, and we much regret that it was not suppressed altogether. It is not only inaccurate, but quite inadequate. A few perfunctory remarks on page 110 are all there is to suggest that it is in this department that the Theory of Sets has achieved its greatest triumphs and has proved itself such a potent agent in its wide applications, not, as Fraenkel suggests, only in "certain" branches of mathematics, but in almost all the branches of thought where mathematical reasoning is applicable.† Here was the place to chronicle, at least by name, some of those simple but brilliant theorems due not so much to Cantor as to other and later workers in this field, which have become part of the every-day equipment of the modern mathematician, such as the Heine-Borel Theorem and the Theorem of Baire. The fruitfulness of this part of the field has been due at least in part to the fact that the idea of *order* does not enter; indeed, in spite of the energy which has been expended on the theory of ordered sets, it remains for the most part a mere pastime. Yet this chapter is presented in the light of a side-issue in the theory of ordered sets.

The book, regarded as a literary whole, is indeed properly an introduction to the idea of the infinite *in the ordinal sense*, that is to say, as Aristotle puts it, when we contemplate the world around us from the point of view of a series, as, for example, when the notion of time is sub-consciously present. Aristotle, however, had grasped the fact that such a point of view is not always admissible, and it is one of the most interesting, as well as one of the earliest achievements of Georg Cantor that he demonstrated the existence of two distinct infinite magnitudes in the realm of the *cardinally* infinite. Fraenkel's presentation of this part of the subject is open to criticism. On page 7 a paragraph has been introduced connecting the example given of the infinite set, say A, consisting of the points whose distances from the origin are $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots 2^{-n}, \ldots$ with Zeno's problem of "Achilles and the tortoise." The fallacy contained in the paradoxical presentation of this problem was completely explained by Aristotle, who seems to have reached the notion of the so-called *improper infinite*, though he and his successors very reasonably denied to it the property of magnitude. It has not been pointed out that the underlying notion that a definite and distinctive character can be connected with this and other infinites is due to the fact, noted by Fraenkel, that the infinite set A is "thoroughly comprehensible and logically unassailable," our minds being capable of conceiving it as a whole, *apart from its serial character*. The symmetrical form or εἶδος enables us to grasp the set of points

* We remark that Georg Cantor (a Jew, born in Russia), is no longer called a "German mathematician," but referred to Halle, at which German university he lived and died professor. The distinction is a neat one: the Pan-German claim that the geniuses of the world are all or mostly "Germans" is as untenable as the claim of intellectual Germany to be the foster-mother of European intellect is undeniable.

† See Professor W. H. Young's Presidential Address to the London Mathematical Society November 1925.

A more easily than the finite subset consisting, for instance, of all those points 2^{-n} for which n is less than a million.

It is perhaps due to the same faculty of our minds that the Infinitesimal Calculus has introduced a simplification into our treatment of problems in the Applied Sciences, and not a complication. In fact we again grasp more easily the appearance of the set, say C, of all the points of a straight line whose distances from the origin are less than unity, than that of all of these points whose distances from the origin are also greater than the limit of vision. But the reduction of the geometrical notion of a straight line and its points to an arithmetical one was only rendered finally clear by the work of Richard Dedekind from one point of view and of Georg Cantor from another, and this is inadequately presented by Fraenkel on page 8 of his book. Even the great mathematician Felix Klein did not explain correctly the double fact that (i) to each point of a straight line there may be made to correspond one and only one non-terminating decimal fraction, while (ii) to each such decimal there may be made to correspond one and only one point of the straight line. Klein failed to bring out the distinction that, whereas the first statement depends only on the meaning of the terms used, the second requires the Cantor-Dedekind Axiom, to which, in fact, its acceptance is logically equivalent.

The introduction by Fraenkel of *the arithmetic continuum* is insidious, and renders doubtful its use in the later parts of the book (pp. 25, 34-37). The attitude of Kronecker and other arithmeticians in denying the property of "existence" as numbers to non-terminating decimal expressions, unless at least a clear and definitely worded law could be given for the filling of the places by the digits 0, 1, . . . 9, is due precisely to their refusal to recognise the Cantor-Dedekind Axiom, and is an attitude as logically unassailable as that of their opponents.

The formal proof given on pp. 34-36 of the distinct natures of the infinite number a (which is that connected with the enumeration of the set above called A) and the number c (connected with the set above called C) requires revision. The proof originally given by Cantor involved the use of continued fractions, and is perfect. The alternative use of decimals requires some care, owing to the fact that a number has two modes of expression as a decimal if, and only if, one of these is terminating (that is, ends in a circulating $\dot{0}$), or ends in a circulating $\dot{9}$. Though Professor Fraenkel has attempted in the small print on p. 36 to fill in one of these pitfalls, he has overlooked the other. His proof* might be materially shortened and at the same time rendered valid, by proceeding to form from the set, say D, of decimal fractions

$$0 . a_1a_2a_3 \ldots,$$
$$0 . b_1b_2b_3 \ldots, \text{ etc.},$$

the perfectly determinate decimal

$$d = 0 . \alpha_1\beta_2\gamma_3 \ldots,$$

where α_1 is the first of the digits 1, 2, . . . 8 different from a_1, β_2 from b_2, and so on. Since the number d represented has no second mode of representation as a decimal, and this decimal itself is formally different from each of the members of the set D, it is clear that we have here a number d determined by the set D, and not contained in that set. But by a proper choice of the a's and b's, etc., the set D can be made to represent *any* set which can be enumerated *à la mode A*, that is any "countable" set, as we say: this proves, therefore, that no countable set can exhaust the whole continuum of decimal fractions, so that the infinite number a which characterises a set as *countable* must be different from the number c which plays the same rôle with respect to the continuum.

The very dubious remark on p. 37 that "the reader must not let himself be worried if he finds it necessary to go through the proof over and over again until it seems to him self-evident" will then be superfluous.

A careful study of Fraenkel's book might be of excellent service to one willing and able to weigh what is there given in the light of the original

* Which was that given in the first edition of Young's *Theory of Sets and Points*, and is being emended in the forthcoming second edition.

writings and their modern developments. We will give one instance here of what is meant. The definition of the term "everywhere dense" on page 106 is wrong, or, more correctly, misleading, since in an ordinal sense it may be said to hold as an extension of the term as habitually used. This term was introduced by Cantor, and to suppress this in *a soi-disant* introduction to the theory he founded, and define the term in a different sense, cannot be justified. The idea of order (p. 105) in this part of the subject is, as already remarked, an anomalous feature in Fraenkel's presentation. He has apparently, though tacitly, moulded it here on Hausdorf's paper of 1908 in the *Mathematischen Annalen* (vol. lxv.) entitled "Grundzüge einer Theorie der geordneten Mengen," where, however, no misconception is possible. In Hausdorf's book on the Theory of Sets (1914) the matter is put in its appropriate place with reference to the whole subject, and the classical definition is given in its proper connexion. Fraenkel replaces *ab initio* the straight line by the ordered arithmetic continuum, that is by an ordered set M corresponding to the decimal fractions *in order of magnitude*, and he confines his attention to sub-sets of M *ordered in the same manner*. Now the geometrical straight line does not, in itself, involve primarily any determinate notion of order; moreover, when such an order is superimposed, it is frequently used only *in a differential sense*, that is confining our attention to the neighbourhood of the points actually discussed. The linear sets of points which occur in practice, when an idea of order is connected with them at all, are more frequently ordered in some mode different from that of the decimal fractions. In all cases, however, the notion of "dense everywhere" and that of "dense in itself" apply and are useful. *Everywhere* implies reference to the *underlying continuum** regarded, not as a set of points, but as a whole divisible into parts similar to itself, that is into *partial intervals*,† which, as such, have no pre-determined order. The difficulty in passing from this analytic and geometric conception of the straight line to the synthetic arithmetic one of the continuum as consisting of points, was one that haunted the ancient Greeks, as we see in particular from Aristotle's tract on "Atomic Lines" ‡

The idea of *dense everywhere* is now very obvious. A set of points is said to be *dense everywhere*, when in every partial interval there lies at least one point of the set.

The definition given by Fraenkel (p. 106) is, however, as follows :—"A set of points N is said to be everywhere dense, or shortly dense, when between every two of its points there is a further point of the set N." In other words: In every partial interval *whose end-points belong to the set N* there are at least three points of the set.

Thus, according to the accepted definition of Cantor, the rational points—that is those points corresponding to the fraction m/n, where m and n are integers prime to one another—form a set dense everywhere. This is in accordance with ordinary common sense; the term "everywhere" would clearly have no meaning if we asserted that an example of a set dense everywhere was given by all the rational points except those in the closed interval

$$\tfrac{1}{2} \leqslant m/n \leqslant \tfrac{3}{4}.$$

Yet we should have to commit this absurdity if we accepted Fraenkel's definition.

Again, the presence of an isolated point will not necessarily overthrow the property required by Fraenkel. For instance, we may add to the set just cited the isolated point $\frac{5}{8}$ without vitiating the property. Thus a set which satisfies Fraenkel's requirements is not even "dense in itself." The term *dense in itself*, also due to Cantor, implies that no point of the set is isolated, that is each point P of the set can be approached indefinitely as limit by points of the set.

On p. 111 Fraenkel gives a definition of "dense in itself" as follows :—"A set of points N is said to be dense in itself, if for every one of its points

* Or *set* in more general work.

† The word *intervals* must in the general case be understood to mean *subsets suitably defined.*

‡ Tr. by H. Joachim in the Oxford edition of the *Works of Aristotle*.

P the following condition is fulfilled : supposing P to lie between any pair of points P_1 and P_2 of N, then there lies also another point Q of N between P_1 and P_2."

It is not clear from the wording of this definition whether Fraenkel demands in it that each point P should lie between other points of the set. Even so, the set is not necessarily dense in itself in the usual sense, since, as before, it might contain isolated points. On the other hand, if Fraenkel does not mean to make this demand, there is no restriction on the end-points of the set, so that, for instance, the open interval

$$1 < x < 2,$$

together with the isolated point $x=0$, satisfies Fraenkel's definition, but is not *dense in itself* in the accepted sense. Here again Fraenkel has been misled by his desire to make everything depend upon *order*.

In spite of these and similar blemishes, this little book has, we hope, a future. There is much in it which is well put, and it has the merit of being exceedingly readable. The attempt to popularise a subject which holds an important place in the mathematics of to-day is certainly one to be supported.

GRACE CHISHOLM YOUNG.

Einführung in die Analytische Geometrie. By A. SCHOENFLIES. Pp. 304. 15 Goldmark (16·50 Bound). 1925. (Springer, Berlin.)

This is Volume XXI. of the Grundlehren der Mathematischen Wissenschaften series which has been running for only two or three years, but has justified its existence by containing several interesting and important works.

The author of the present book, who is Professor of Mathematics in the University of Frankfurt, explains the scope of the book in his very first sentence, where he remarks that the gap between an ordinary school course of analytical geometry and the higher levels of the subject is perhaps greater than the corresponding gap in any other part of mathematics. The book is therefore intended to satisfy a definite need ; and without overburdening the reader with a wealth of detail, as it must be confessed is usually the case in English text-books on geometry, the author carries his thesis far enough to give a very good idea of algebraic geometry as it stands nowadays.

A review of the contents is worth giving, as it shows the ground covered and the arrangement of subject matter. Beginning with an interesting elementary account of the arithmetical and geometrical continuum, the author passes on in Chapter 2 to explain all ordinary coordinate systems, including bi-angular and bi-polar systems. He shows very clearly how these are all part of the same general scheme. Chapter 3 illustrates this with circles, conics, straight lines, and the spiral of Archimedes (in this order), all treated very directly and simply. Chapters 4 and 5 deal systematically with the straight line in Cartesian coordinates. Chapter 6 introduces line coordinates with the principle of duality, and fundamental linear theorems of projective geometry. Chapter 7 has cross ratios, defined as a double ratio, also involution, related ranges and pencils, and Chasles' double theorem for the description of a conic by a moving point or a moving line. Then, in Chapter 8, come homogeneous coordinates, the general linear transformation and its connexion with projection. Chapters 9 and 10 deal with the circle, and conics referred to their axes. Chapter 11 discusses the general equation of the second degree, and Chapter 12 includes a short but clear account of collineations and reciprocations, ending with a valuable three-page (180-182) historical note. Five chapters on three-dimensional geometry follow, and they cover the corresponding ground. This association of plane and solid analytical geometry in one book is much to be recommended. Then a few detached notes on determinants, matrices, imaginary numbers and magnitudes, and a collection of examples.

The book has been written with a breadth of outlook, by one who has not forgotten that he is writing to enlighten the reader who may happen not to be deeply versed in geometry or in analysis. It is a good supplement to existing books in English, and is very well worth reading by all teachers at school or college who deal with analytical geometry, some of whom have

other mathematical interests, but would like to know more exactly how this geometry fits in with the general scheme of things mathematical.

The book is well printed, with an adequate index and plenty of suitable geometrical figures.

A small point is worth adding. The book is an excellent handbook of technical terms—geometric and algebraic—some of which even to-day by no means household words even among mathematicians. What, for example, is the English for *Übertragungsprinzipien*, a name given to an algebraic process due to Clebsch, of importance in projective geometry ?

The book would be more useful still if it gave more specific references to original works and to existing larger treatises to which it forms a good introduction.

Cours de Mathématiques Générales, *à l'usage des étudiants en Sciences Naturelles*. By G. Verriest. Part II. Pp. 1-388. 38 francs. 1925. (Louvain, Éditions Universitas ; Paris, Gauthier-Villars.)

This volume deals with Analytical Geometry of three dimensions, and the Integral Calculus. Although it is self-contained, it has passing references to the preceding volume, which it naturally follows. The whole work is written primarily for students reading for a Doctorate in Chemistry at Louvain, and more generally for others to whom mathematics is an interest or a necessity.

There are few departments of Natural Science which mathematics does not illumine, and in this age of intensely narrow specialising, it is strange how the need of a broader training becomes more and more insistent. Apart from the well-known applications of mathematics to Natural Philosophy in general, the calls upon it in the realm of chemistry and of medicine are becoming quite bewildering. There is an eminent zoologist also who holds * that a knowledge of conformal representation in the theory of complex variables throws light on the growth of a lobster or a vegetable marrow, or even on a Dürer engraving. One is told by the organic chemists that a rich field of research awaits the student who is master of chemistry and of pure geometry, especially of space filling figures. This too at a time when the study of pure geometry is at a low ebb, although happily of late there have been distinct signs of the returning flow.

Undoubtedly there is every inducement for Applied Mathematics to enlarge its borders as much as it wills. Yet at the same time the ever continuing advances in Pure Mathematics make the gap, between mathematics and the sciences in general, more difficult to bridge. For the delicate work of relaying the foundations, let us say, of the differential calculus and of geometry has left its mark ; and we mathematicians now run the risk of being thought incapable of admitting that the Mean Value Theorem is true or that there is such a thing as the distance between two points.

The first volume of this present book was very favourably reviewed in the *Gazette* a few years ago (Jan. 1924, p. 25). This second volume merits equal praise. It is a masterly exposition, on elementary lines, of the Integral Calculus, written with a rare clarity and charm. It covers the ground which must be traversed by the college student who reads enough mathematics to make a useful tool in his physics or chemistry. It would also give an excellent preliminary course for the student who wishes to go deeply into mathematical analysis.

The treatment is frankly geometrical, but the existence of rigid arithmetical proofs is occasionally acknowledged. The short account of series, given in the appended Note iv. pp. 355-375, is extremely good.

The actual details of contents call for little further comment. There is just enough Cartesian three-dimensional geometry, in sixty pages, to meet the needs of later problems on lengths of twisted curves, areas of surfaces and the like. Then comes the integral calculus systematically developed, and finally some useful notes on interpolation and allied problems.

The choice of apt illustrations from a very wide range of physical subjects is interesting. H. W. Turnbull.

* Cf. *Trans. Edinburgh Roy. Soc.* vol. l. (1915).

Linaludo—The Knight's Tours. Instructions and Sketch Books. By A. SHARP. 4s. or 3s. + 9d. + 9d. 1926. (E. Marlborough & Co.)

A pastime on the theory of which mathematicians so eminent as De Moivre, Euler and Vandermonde have not disdained to employ their powers may fairly claim some notice in the *Gazette*. Of the above booklets, *Instructions*, treating the subject as a game of Patience, contains explanations of terms and methods. It discusses tours with a centre or with one, two, or three axes; of symmetry, and shows how sets of partial tours may be transformed into a "grand tour." The first 17 pages are restricted to tours on an ordinary 8×8 board; in the remaining 15 pages this restriction is removed. *Sketch Books* 1 *and* 2 serve for trials at tours suggested in *Instructions* or for records of success. No. 1 gives 128 skeleton diagrams each containing 64 small rings each arranged in 8 rows and 8 columns, the centre of each ring indicating that of a square on a chessboard, black by single, white by double rings. The author suggests that first attempts should be made on 6-square diagrams, the full 8-square tours not being tried until after some skill in minor operations has been acquired. It seems to us that at first restriction might be made to what may be called the *accidence* of the subject, that is the practice of solutions of the simple problems of reducing *stops* $m + ni$ (vector notation), m and n being positive or negative integers numerically less than 8; to a concatenation of Knight's moves, *e.g.* the simplest cases:

$$1 = 1 \pm 2i - (2 \pm i) + 2 \mp i),$$

$$i = \pm 2 + i + (\mp 1 + 2i) - (\pm 1 + 2i),$$

$$1 + 1 = 2 - i + (2i - 1),$$

with repeated practice in passing from diagram to arithmetical analysis and *vice versâ*. No. 2 gives 32 pages each of 616 such rings arranged in 22 columns and 28 rows. Both sketch books may be used for trials and records of other chessboard problems, *e.g.* Sprague's *Non-linear arrangements of eight men on a chessboard* (*Proc. Ed. Math. Soc.*, viii.) and Cotter's chessboard proofs of the rectangular property of triangles whose sides are given by $m^2 + n^2$, $m^2 - n^2$ and $2mn$ (*Nature*, 6/v./22). We have tested No. 2 with several examples of Mr. Cotter's diagram and have found the spacing of its rings very accurate.

Exercises in Geometry. Part II. By V. LE NEVE FOSTER. Pp. i-viii + 144. 2s. 9d. 1925. (Bell & Sons.)

This collection is of the same general character as Part I. reviewed in No. 180 of the *Gazette*: the range extending to Stages II. and III. of the author's scheme, while Part I. was limited to Stage I. It has the same merits—apt advice on method, great variety in the selection, excellent diagrams. It is a storehouse of things new and old well suited for exercising the skill and stimulating the imagination of the student who uses it.

EDWARD M. LANGLEY.

Primer of Arithmetic for Middle Forms. By F. M. MARZIALS and N. K. BARBER. Pp. xii + 262. 3s. 6d. net. 1925. (Oxford Univ. Press.)

The purpose of this book is accurately described by its title.

Defining Arithmetic as "The Science of Computation," the authors have kept strictly to this definition, excluding Graphs, but including a short section on the Trigonometrical Ratios. The general plan of the book is admirable, and the authors are justified in stating that the book can be read by the pupil with the minimum of assistance; the examples are good, sufficient, and well graded.

Very few teachers will agree with the treatment of multiplication and division of decimals. It is arguable that the old methods of "counting up the number of digits after the decimal point" and "making the divisor a whole number" may be the easiest to operate (though even this is doubtful), but "standard form" has come to stay, and it is to be regretted that such an excellent book as this should revert to the old methods.

Exercises in Algebra. From the Beginnings to the Quadratic. By R. W. M. GIBBS. Pp. 160. 1s. 6d. net. 1925. (Oxford Univ. Press.)

A most stimulating introduction to Algebra. No explanations are given, but the bookwork needed for each section is tabulated at the beginning of the section, and the author's line of thought is very clearly indicated by the examples, which are progressive, varied, and in many instances refreshingly original. It is perhaps unfortunate that the word "cancel" is used in connection with fractions, and it is contrary to general usage to treat "factor" as a verb, but these are minor flaws, and the book as a whole is thoroughly to be commended to teachers of young boys and girls. N. J. C.

Mechanics and Applied Mathematics. Dynamics, Statics, Hydrostatics. By W. D. HILLS, B.Sc. Part I. Mechanics. Pp. 250 + xi. 4s. net. (London University Press.)

Part I. of this book covers the ground of the London Matriculation syllabus. It is written for students of science and Engineering whose mathematical equipment may be small.

There is an unfortunate illustration of "action and reaction" on p. 4, and a few lapses occur elsewhere, but on the whole the treatment is good. There are many excellent diagrams, a useful summary at the end of each chapter, and a total of about 350 examples with answers. The book should prove useful to a considerable class of students. A. R.

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366. Puzzled Commissioners.—Mr. Gore was recalled, and put in a number of tracings of designs which, he said, he deposited with the Admiralty. The Commission was occupied for a long time trying to get counsel to explain many terms used in the designs of motors. The witness described in scientific language one design as a tetrahedron.

Mr. Justice Tomlin—What is meant by a tetrahedron as now used?

The witness explained that the design presented a wedge whichever way it was looked at.

Why does that make it a tetrahedron? asked Mr. Justice Tomlin, adding that his own Greek did not seem to meet this particular case. (Laughter.)

Witness borrowed a sheet of writing paper, and folded it to represent the bow of a boat.

Mr. Justice Tomlin suggested that it looked more like a trireme, and another Commissioner observed that they all seemed to be getting a little mixed.

At this moment Mr. Trevor Watson was busily engaged in folding a piece of paper.

Mr. Justice Tomlin—Mr. Trevor Watson seems to be reaching the stage of understanding. (Laughter.)

Mr. Watson's attempt was handed to witness, who was asked whether it was a tetrahedron.

Witness (emphatically)—No. (Laughter.) "It is not far off," added the witness after further consideration.

Mr. Justice Tomlin—Not bad for a first attempt. (Laughter.)—From *The Scotsman*, Feb. 10, 1925 (per Mr. C. Tweedie).

367. [D'Alembert tells us that the (Jansenist) teachers at the Collège des Quatre Nations] did their best to dissuade him from mathematics and poetry, alleging that the former, in particular, *dried up the heart*, and recommending, as to the latter, that he should confine himself to the poem of St. Prosper upon Grace.—*D'Alembert*, De Morgan. *Penny Cyclopaedia.*

368. Charles Boyle, second son of the second Earl of Orrery, was a patron of George Graham, an ingenious watchmaker, who constructed an instrument representing the revolutions of the planets. To this Graham gave in gratitude the name of Orrery. But Dr. Johnson says:—"The whole merit of inventing it belongs to Rowley, a mathematician of Litchfield."

YORKSHIRE BRANCH.

At a meeting of the Yorkshire branch of the Mathematical Association, held at the Leeds University on Saturday, 13th February, Mr. A. B. Oldfield, of Pudsey, was elected secretary in place of Miss H. N. Stephen, who has left the district.

Professor Brodetsky, who was in the chair, reported on the arrangements for the Newton bi-centenary celebrations to be held in March of next year.

Professor Milne gave an interesting paper on "The History of the Equation," and a further discussion on Mr. W. Peaker's paper on "School Certificate Mathematics—Aims and Results," which was given at the last meeting, was introduced by Mr. S. H. Stelfox, H.M.I.

UNIVERSITY OF MANCHESTER.

The Third Session of the Summer Courses in Post-Graduate Mathematics to be held at University College, Aberystwyth, from Thursday, 12th August, to Wednesday, 25th August, 1926. The following three alternative courses, each of 20 one-hour lectures, are proposed:

(*a*) "The Theory of Relativity," by Professor Sydney Chapman, M.A., D.Sc., F.R.S., Imperial College of Science, London.

(*b*) "Elliptic Functions," by Professor L. J. Mordell, M.Sc., F.R.S., Manchester University.

(*c*) "Higher Plane Curves," by Mr. H. W. Richmond, M.A., F.R.S., King's College, Cambridge.

For further information apply to Miss D. Withington, The University, Manchester.

ST. ANDREWS MATHEMATICAL COLLOQUIUM, 1926.

Under the auspices of the Edinburgh Mathematical Society, a Mathematical Colloquium will be held in St. Andrews from 3rd August to 13th August, 1926. The following courses of lectures have been arranged:

(*a*) "The Significance of Dynamics for Scientific Thought," by George D. Birkhoff, Ph.D.

(*b*) "Recent Developments in Algebraic Geometry," by H. W. Richmond, LL.D., F.R.S.

(*c*) "Recent Developments in Applied Mathematics (Gravitation)," by S. Brodetsky, D.Sc.

(*d*) "The History of Mathematics in Scotland," by G. A. Gibson, M.A., LL.D., F.R.S.E.

(*e*) "An Introduction to the Invariant Theory," by H. W. Turnbull, M.A., F.R.S.E.

(*f*) Informal Talks will, it is expected, be given by Professor E. T. Whittaker, F.R.S., and others.

For further information apply to E. T. Copson, M.A., The University, Edinburgh.

THE LIBRARY.

160 CASTLE HILL, READING.

ADDITIONS.

The Librarian reports the following gifts :

From Mr. **C. W. Adams**, a collection of pamphlets, including the Final Report of the American National Committee of Fifteen on Geometry Syllabus, and C.I.E.M. publications Nos. 6 and 133-140

From Mr. **F. Robbins**:

D. LARDNER Arithmetic - - - - (Cabinet Cyclopædia 55) 1834

E. S. LOOMIS Original Investigation {1 (1901) rep.} - - - 1902

W. WHEWELL Mechanical Euclid {4} - - - - - - - 1843

By exchange has been acquired :

C. HUYGENS Œuvres ; II. - - - - - - - - - - 1889

We have now eight volumes of the Edition Définitive, namely 1-2, 4, 8-11, 14, and members should note as an example of what can be done by cooperation that these have come to us from four distinct sources.

The following have been bought :

R. D. CARMICHAEL Relativity - - - - - (Merriman-Woodward 12) 1913

Out of date, and bought because of the series to which it belongs, a careful account of the special theory is nevertheless worth reading.

L. A. SOHNCKE Bibliotheca Mathematica - - - - - - 1854

The catalogue of books published between 1830 and 1854 is for purposes of reference, but while he can not send it out, the Librarian will be pleased to consult it to answer any queries which members may raise concerning books of the period. Rogg's work of 1830 with the same title, to which this is a sequel, would be very welcome if any member could give it.

JOURNAL OF THE MATHEMATICAL ASSOCIATION OF JAPAN FOR SECONDARY EDUCATION.

Prof. Hayashi has extended his kindness, reported last year, by finding and giving to us the number which was then lacking, and the Library has now a set of this journal complete from the beginning in 1919.

CATALOGUE OF THE LIBRARY.

WITH this number of the *Gazette* members should receive a catalogue of the Library as it was at the end of 1923. That the list is so far out of date is due to the pains that have been taken with typographical details; the necessary experiments, in which the printers cooperated most heartily, could not have been made on a larger scale, but now that satisfactory forms have been found for the various entires, future lists can be printed more expeditiously. Thanks chiefly to Prof. Genese, to Mrs. Godfrey, and to Mr. Greenstreet, the Library has doubled since 1923, and a second catalogue as large as the first is already in preparation ; members can discover in the pages of volume xii of the *Gazette* the material of which this next list will be composed.

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LOCAL BRANCHES.

The names and addresses of the Honorary Secretaries of the local Branches of the Association are as follows:—

LONDON:	Miss L. A. ZELENSKY, Haberdashers' Aske's Girls' School, Creffield Road, Acton, W. 3.
	G. GOODWILL, Shaw Wood, Farnham Green, Bishop's Stortford, Herts.
YORKSHIRE:	A. B. OLDFIELD, The Secondary School, Pudsey, near Leeds.
BRISTOL:	R. C. FAWDRY, Clifton College, Bristol.
MANCHESTER AND DISTRICT:	Miss W. GARNER, Whalley Range High School.
	G. B. JACKSON, Secondary School, Ashton-under-Lyne.
MIDLAND:	Miss E. M. READ, The Secondary School, King's Norton.
	C. G. NOBBS, King Edward's High School, New Street, Birmingham.
NORTH WALES:	E. R. HAMILTON, University College of North Wales, Bangor.
CARDIFF:	H. A. HAYDEN, University College, Cardiff.
SYDNEY (N.S.W.):	Miss JANET BROWN, Boys' High School, Fort Street, Sydney.
	H. J. MELDRUM, The Teachers' College, Sydney.
QUEENSLAND:	J. P. M'CARTHY, Boys' Grammar School, Gregory Terrace, Brisbane.
VICTORIA:	R. J. A. BARNARD, 21 Bambra Road, Caulfield, Melbourne.
	J. L. GRIFFITH, 1032 Drummond Street, North Carlton, Melbourne.

THE MATHEMATICAL ASSOCIATION.

(An Association of Teachers and Students of Elementary Mathematics.)

"*I hold every man a debtor to his profession; from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves, by way of amends, to be a help and an ornament thereunto.*"—BACON (Preface, *Maxims of Law*).

THE MATHEMATICAL ASSOCIATION, which was founded in 1871, as the *Association for the Improvement of Geometrical Teaching*, aims not only at the promotion of its original object, but at bringing within its purview all branches of elementary mathematics.

Its purpose is to form a strong combination of all persons who are interested in promoting good methods of teaching mathematics. The Association has already been largely successful in this direction. It has become a recognised authority in its own department, and is continuing to exert an important influence on methods of examination.

The Annual Meeting of the Association is held in January. Other Meetings are held when desired. At these Meetings papers on elementary mathematics are read and discussed.

Branches of the Association have been formed in London, Bangor, Yorkshire, Bristol, Manchester, Cardiff, the Midlands (Birmingham), New South Wales (Sydney), Queensland (Brisbane), and Victoria (Melbourne). Further information concerning these branches can be obtained from the Honorary Secretaries of the Association.

"**The Mathematical Gazette**" (published by Messrs. G. BELL & SONS, LTD.) is the organ of the Association. It is issued at least six times a year. The price per copy (to non-members) is usually 2s. 6d. each. The *Gazette* contains—

(1) ARTICLES, mainly on subjects within the scope of elementary mathematics;

(2) NOTES, generally with reference to shorter and more elegant methods than those in current text-books;

(3) REVIEWS, written when possible by men of eminence in the subject of which they treat. They deal with the more important English and Foreign publications, and their aim is to dwell on the general development of the subject, as well as upon the part played therein by the book under notice;

(4) QUERIES AND ANSWERS, on mathematical topics of a general character.

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